Lecture notes on Topology

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ABSTRACT. This is a set of lecture notes prepared for a series of introductory courses in Topology for undergraduate students at the University of Science, Vietnam National University–Ho Chi Minh City. It is written to be delivered by a lecturer, namely by myself, tailored to the need of my own students. I did not write it with self-study readers or other lecturers in mind.

In my experience many things here are much better explained in oral form than in written form. Therefore in writing these notes I intend that more explanations and discussions will be carried out in class. I hope by presenting only the essentials these notes will be more suitable for classroom use. Some details are left for students to fill in or to be discussed in class.

Since students in my department are required to take a course in Functional Analysis, I try not to duplicate material in that course.

A sign * in front of a problem notifies the reader that this is an important problem (which clarifies understanding or will be used later) although might not appear to be so at first, so the reader should try it.

This set of lecture notes will be continuously developed. The latest version is available on my web page [http://www.math.hcmus.edu.vn/~hqvu](http://www.math.hcmus.edu.vn/~hqvu)

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CHAPTER 1

General Topology

1.1. Set

In General Topology we often work in very general settings, in particular we often deal with infinite sets.

We will not define what a set is. That means we only work on the level of the so-called “naive set theory”. Even so we should be aware of certain problems in naive set theory. Until the beginning of the 20th century, the set theory of George Cantor, in which set is not defined, was thought to be a good basis for mathematics. Then some critical problems were discovered.

Example 1.1.1 (Russell’s paradox). Consider the set 
\[ S = \{ x \mid x \not\in x \} \]
(the set of all sets which are not members of themselves). Then whether \( S \in S \) or not is undecidable, because answering yes or no to this question leads to contradiction. \[\square\]

Axiomatic systems for the theory of sets have been developed since then. In the Von Neumann-Bernays-Godel system a more general notion than set, called class (lớp), is used, see \[\text{Dug66}, \text{p. 32}\]. In this course, in occasions where we deal with “set of sets” we often replace the term set by the terms class or collection (họ).

Indexed collection. Suppose that \( A \) is a collection, \( I \) is a set and \( f : I \to A \) is a surjective map. The map \( f \) is called an indexed collection, or indexed family (họ được đánh chỉ số). We often write \( f_i = f(i) \), and denote the indexed collection \( f \) by \( \{ f_i \}_{i \in I} \). Notice that it can happen that \( f_i = f_j \) for some \( i \neq j \).

Thus an indexed collection is a map, not a collection.

Relation. A relation (quan hệ) \( R \) on a set \( S \) is a non-empty subset of the set \( S \times S \). When \( (a, b) \in R \) we often say that \( a \) is related to \( b \), and write \( aRb \).

A relation is:

1. reflexive (phản xạ) if \( \forall a \in S, aRa \).
2. symmetric (đối xứng) if \( \forall a, b \in S, aRb \Rightarrow bRa \).
3. antisymmetric (phán đối xứng) if \( \forall a, b \in S, ((aRb) \land (bRa)) \Rightarrow a = b \).
4. transitive (bắc cầu) if \( \forall a, b, c \in S, ((aRb) \land (bRc)) \Rightarrow (aRc) \).

\[\text{1} \text{Discovered in 1901 by Bertrand Russell. A famous version of this paradox is the barber paradox: In a village there is a barber; his job is to do hair cut for a villager if and only if the villager does not cut his hair himself. If we take the set of all villagers who had their hairs cut by the barber then we are not able to decide whether the barber himself is a member of that set or not.}\]
An equivalence relation on $S$ is a relation that is reflexive, symmetric and transitive.

If $R$ is an equivalence relation on $S$ then an equivalence class (lớp tương đương) represented by $a \in S$ is the subset $[a] = \{ b \in S \mid aRb \}$. Two equivalence classes are either coincident or disjoint. The set $S$ is partitioned (phân hoạch) into the disjoint union of its equivalence classes.

Equivalent sets. Two sets are said to be equivalent if there is a bijection from one to the other.

**Example 1.1.2.** Two intervals $[a, b]$ and $[c, d]$ on the real number line are equivalent. Similarly, two intervals $(a, b)$ and $(c, d)$ are equivalent. The bijection can be given by a linear map $x \mapsto \frac{d-c}{b-a}(x-a) + c$.

The interval $(-1, 1)$ is equivalent to $\mathbb{R}$ via a map related to the tan function:

$$x \mapsto \frac{x}{\sqrt{1-x^2}}.$$

**Figure 1.1.1.** The interval $(-1, 1)$ is equivalent to the line $\mathbb{R}$ as sets.

Countable sets.

**Definition 1.1.3.** A set is called countably infinite (vô hạn đếm được) if it is equivalent to the set of all positive integers. A set is called countable if it is either finite or countably infinite.

Intuitively, a countably infinite set can be “counted” by the positive integers. The elements of such a set can be indexed by the positive integers as $a_1, a_2, a_3, \ldots$.

**Example 1.1.4.** The set $\mathbb{Z}$ of all integer numbers is countable.

The set of all even integers is countable.

**Proposition 1.1.5.** A subset of a countable set is countable.

**Proof.** The statement is equivalent to the statement that a subset of $\mathbb{Z}^+$ is countable. Suppose that $A$ is an infinite subset of $\mathbb{Z}^+$. Let $a_1$ be the smallest number in $A$. Let $a_n$ be the smallest number in $A \setminus \{a_1, a_2, \ldots, a_{n-1}\}$. Then $a_{n-1} < a_n$ and the set $B = \{a_n \mid n \in \mathbb{Z}^+\}$ is a countably infinite subset of $A$.

We show that any element $m$ of $A$ is an $a_n$ for some $n$, and therefore $B = A$. 
Let \( C = \{ a_n \mid a_n \geq m \} \). Then \( C \neq \emptyset \) since \( B \) is infinite. Let \( a_{n_0} = \min C \). Then \( a_{n_0} \geq m \). Further, since \( a_{n_0-1} < a_{n_0} \) we have \( a_{n_0-1} < m \). This implies \( m \in A \setminus \{ a_1, a_2, \ldots, a_{n_0-1} \} \). Since \( a_{n_0} = \min (A \setminus \{ a_1, a_2, \ldots, a_{n_0-1} \}) \) we must have \( a_{n_0} \leq m \). Thus \( a_{n_0} = m \). □

**Proposition 1.1.6.** If there is a surjective map from \( \mathbb{Z}^+ \) to a set \( S \) then \( S \) is countable.

**Proof.** Suppose that there is a surjective map \( \phi : \mathbb{Z}^+ \to S \). For each \( s \in S \) the set \( \phi^{-1}(s) \) is non-empty. Let \( n_s = \min \phi^{-1}(s) \). The map \( s \mapsto n_s \) is an injective map from \( S \) to a subset of \( \mathbb{Z}^+ \), therefore \( S \) is countable, by 1.1.5. □

**Theorem 1.1.7.** The union of a countable collection of countable sets is a countable set.

**Proof.** The collection can be indexed as \( A_1, A_2, \ldots, A_i, \ldots \) (if the collection is finite we can let \( A_i \) be the same set for all \( i \) starting from a certain index). The elements of each set \( A_i \) can be indexed as \( a_{i,1}, a_{i,2}, \ldots, a_{i,j}, \ldots \) (if \( A_i \) is finite we can let \( a_{i,j} \) be the same element for all \( j \) starting from a certain index).

This means there is a surjective map from the index set \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) to the union \( \bigcup_{i \in I} A_i \) by \( (i,j) \mapsto a_{i,j} \).

Thus it is sufficient for us, by 1.1.6, to prove that \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) is countable.

We can index \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) by the method shown in the following diagram:

\[
\begin{array}{ccccc}
(1,1) & \rightarrow & (1,2) & \rightarrow & (1,3) \\
| & & | & & |
\downarrow & & \downarrow & & \downarrow \\
(2,1) & (2,2) & (2,3) & & \\
| & | & | & & |
\downarrow & \downarrow & \downarrow & & \\
(3,1) & (3,2) & & & \\
& & & & \\
(4,1) & (4,2) & & & \\
& & & & \\
(5,1) & & & & \\
\end{array}
\]

□

**Theorem 1.1.8.** The set \( \mathbb{Q} \) of all rational numbers is countable.

**Proof.** One way to prove this is by writing \( \mathbb{Q} = \bigcup_{q=1}^{\infty} \{ \frac{p}{q} \mid p \in \mathbb{Z} \} \).

Another way is by observing that if we write each rational number in reduced form \( \frac{p}{q} \) with \( q > 0 \) and \( \gcd(p,q) = 1 \) then the map \( \frac{p}{q} \mapsto (p,q) \) from \( \mathbb{Q} \) to \( \mathbb{Z} \times \mathbb{Z} \) is injective. □

**Theorem 1.1.9.** The set \( \mathbb{R} \) of all real numbers is uncountable.

**Proof.** The proof uses the Cantor diagonal argument.
1. GENERAL TOPOLOGY

Suppose that set of all real numbers in decimal form in the interval \([0, 1]\) is countable, and is enumerated as a sequence \(\{a_i \mid i \in \mathbb{Z}^+\}\). Let us write
\[
a_1 = 0.a_{1,1}a_{1,2}a_{1,3} \ldots
\]
\[
a_2 = 0.a_{2,1}a_{2,2}a_{2,3} \ldots
\]
\[
a_3 = 0.a_{3,1}a_{3,2}a_{3,3} \ldots
\]
\[
\vdots
\]

There are real numbers whose decimal presentations are not unique, such as \(\frac{1}{2} = 0.5000 \ldots = 0.4999 \ldots\). Choose a number \(b = b_1b_2b_3 \ldots\) such that \(b_n \neq 0, 9\) and \(b_n \neq a_n\). Choosing \(b_n\) differing from 0 and 9 will guarantee that \(b \neq a_n\) for all \(n\) (see more at 1.1.31). Thus the number \(b\) is not in the above table, a contradiction. □

**Theorem 1.1.10 (Cantor-Bernstein-Schroeder).** If \(A\) is equivalent to a subset of \(B\) and \(B\) is equivalent to a subset of \(A\) then \(A\) and \(B\) are equivalent.

**Proof.** Suppose that \(f : A \mapsto B\) and \(g : B \mapsto A\) are injective maps. Let \(A_1 = g(B)\), we will show that \(A \sim A_1\).

Let \(A_0 = A\) and \(B_0 = B\). Define \(B_{n+1} = f(A_n)\) and \(A_{n+1} = g(B_n)\). Then \(A_{n+1} \subset A_n\). Furthermore via the map \(g \circ f\) we have \(A_{n+2} \sim A_n\), and \(A_n \setminus A_{n+1} \sim A_{n+1} \setminus A_{n+2}\).

Using the following identities
\[
A = (A \setminus A_1) \cup (A_1 \setminus A_2) \cup \cdots \cup (A_n \setminus A_{n+1}) \cup \cdots \cup \bigcap_{n=1}^\infty A_n,\]
\[
A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup \cdots \cup (A_n \setminus A_{n+1}) \cup \cdots \cup \bigcap_{n=1}^\infty A_n,\]
we see that \(A \sim A_1\). □

**Order.** An order (thứ tự) on a set \(S\) is a relation \(R\) on \(S\) that is reflexive, antisymmetric and transitive.

Note that two arbitrary elements \(a\) and \(b\) do not need to be comparable; that is, the pair \((a, b)\) may not belong to \(R\). For this reason an order is often called a partial order.

When \((a, b) \in R\) we often write \(a \leq b\). When \(a \leq b\) and \(a \neq b\) we write \(a < b\).

If any two elements of \(S\) are related then the order is called a total order (thứ tự toàn phần) and \((S, \leq)\) is called a totally ordered set.

**Example 1.1.11.** The set \(\mathbb{R}\) of all real numbers with the usual order \(\leq\) is totally ordered.
Example 1.1.12. Let $S$ be a set. Denote by $2^S$ the collection of all subsets of $S$. Then $(2^S, \subseteq)$ is a partially ordered set, but is not totally ordered if $S$ has more than one element.

Example 1.1.13 (Dictionary order). Let $(S_1, \leq_1)$ and $(S_2, \leq_2)$ be two ordered sets. The following is an order on $S_1 \times S_2$: $(a_1, b_1) \leq (a_2, b_2)$ if $(a_1 < a_2)$ or $(a_1 = a_2) \land (b_1 \leq b_2))$. This is called the dictionary order (thứ tự từ điển).

In an ordered set, the smallest element (phần tử nhỏ nhất) is the element that is smaller than all other elements. More concisely, if $S$ is an ordered set, the smallest element of $S$ is an element $a \in S$ such that $\forall b \in S, a \leq b$. The smallest element, if exists, is unique.

A minimal element (phần tử cực tiểu) is an element which no element is smaller than. More concisely, a minimal element of $S$ is an element $a \in S$ such that $\forall b \in S, b \leq a \Rightarrow b = a$. A minimal element, if exists, may not be unique.

A lower bound (chặn dưới) of a subset of an ordered set is an element of the set that is smaller than or equal to any element of the subset. More concisely, if $A \subset S$ then a lower bound of $A$ in $S$ is an element $a \in S$ such that $\forall b \in A, a \leq b$. The definitions of largest element, maximal element, and upper bound are similar.

Cardinality. A genuine definition of cardinality of sets requires an axiomatic treatment of set theory. Here we accept that for each set $A$ there exists an object called its cardinal (lực lượng, bản số) $|A|$, and there is a relation $\leq$ on the set of cardinals such that:

1. If $A$ is finite then its cardinal is its number of elements.
2. Two sets have the same cardinals if and only if they are equivalent:
   \[ |A| = |B| \iff (A \sim B) \]
3. $|A| \leq |B|$ if and only if there is an injective map from $A$ to $B$.

Theorem 1.1.10 says that $(|A| \leq |B| \land |B| \leq |A|) \Rightarrow |A| = |B|$.

The cardinal of $\mathbb{Z}^+$ is denoted by $\aleph_0$, so $|\mathbb{Z}^+| = \aleph_0$. The cardinal of $\mathbb{R}$ is denoted by $c$ (continuum), so $|\mathbb{R}| = c$.

Proposition 1.1.14. $\aleph_0$ is the smallest infinite cardinal, and $\aleph_0 < c$.

Proof. It comes from that any infinite set contains a countably infinite subset, and $|\mathbb{N}| = \aleph_0$.

Theorem 1.1.15 (No maximal cardinal). The cardinal of a set is strictly less than the cardinal of the set of all of its subsets, i.e. $|A| < |2^A|$.

This implies that there is no maximal cardinal. There is no “universal set”, “the set which contains everything”, or “the set of all sets”.

\[^2\text{}\aleph\text{ is read “aleph”, a character in the Hebrew alphabet.}\]
\[^3\text{Georg Cantor put forward the Continuum hypothesis: There is no cardinal between $\aleph_0$ and $c$.}\]
1. GENERAL TOPOLOGY

Let \( A \neq \emptyset \) and denote by \( 2^A \) the set of all of its subsets.

1. \( |A| \leq |2^A| \): The map from \( A \) to \( 2^A \): \( a \mapsto \{a\} \) is injective.
2. \( |A| \neq |2^A| \): Let \( \phi \) be any map from \( A \) to \( 2^A \). Let \( X = \{a \in A \mid a \not\in \phi(a)\} \). Suppose that there is \( x \in A \) such that \( \phi(x) = X \). Then the question whether \( x \) belongs to \( X \) or not is undecidable. Therefore \( \phi \) is not surjective.

\[\square\]

The Axiom of choice.

Theorem 1.1.16. The following statements are equivalent:

1. Axiom of choice: Given a collection of non-empty sets, there is a function defined on this collection, called a choice function, associating each set in the collection with an element of that set.
2. Zorn lemma: If any totally ordered subset of an ordered set \( X \) has an upper bound then \( X \) has a maximal element.

Intuitively, a choice function “chooses” an element from each set in a given collection of non-empty sets. The Axiom of choice allows us to make infinitely many arbitrary choices in order to define a function.\[\]

The Axiom of choice is needed for many important results in mathematics, such as the Tikhonov theorem in Topology, the Hahn-Banach theorem and Banach-Alaoglu theorem in Functional analysis, the existence of a Lebesgue unmeasurable set in Real analysis, …

There are cases where this axiom could be avoided. For example in the proof of \[1.1.6\] we used the well-ordered property of \( \mathbb{Z}^+ \) instead. See for instance \[End77\] p. 151] for further material on this subject.

Zorn lemma is often a convenient form of the Axiom of choice.

Cartesian product. Let \( \{A_i\}_{i \in I} \) be a family of sets indexed by a set \( I \). The Cartesian product (tich Decartes) \( \prod_{i \in I} A_i \) of this family is defined to be the collection of all maps \( a : I \to \bigcup_{i \in I} A_i \) such that if \( i \in I \) then \( a(i) \in A_i \). The statement “the Cartesian product of a family of non-empty sets is non-empty” is therefore equivalent to the Axiom of choice.

An element \( a \) of \( \prod_{i \in I} A_i \) is often denoted by \( (a_i)_{i \in I} \), with \( a_i = a(i) \in A_i \) being the coordinate of index \( i \), in analog to the finite product case.

Problems.

1.1.17. Check that \( (\bigcup_{i \in I} A_i) \cap (\bigcup_{j \in J} B_j) = \bigcup_{i \in I, j \in J} A_i \cap B_j \).

1.1.18. Which of the following formulas are correct?

\[\text{Bertrand Russell said that choosing one shoe from each pair of shoes from an infinite collection of pairs of shoes does not need the Axiom of choice (because in a pair of shoes the left shoe is different from the right one so we can define our choice), but if in a pair of socks the two socks are same, then choosing one sock from each pair of socks from an infinite collection of pairs of socks needs the Axiom of choice.}\]
1.1.20. Let \( f \) be a function. Then:
1. \( f(\bigcup_i A_i) = \bigcup_i f(A_i) \).
2. \( f(\bigcap_i A_i) \subset \bigcap_i f(A_i) \). If \( f \) is injective (one-one) then equality happens.
3. \( f^{-1}(\bigcup_i A_i) = \bigcup_i f^{-1}(A_i) \).
4. \( f^{-1}(\bigcap_i A_i) = \bigcap_i f^{-1}(A_i) \).

1.1.21. A union between a countable set and a finite set is countable.

1.1.22. If \( \pi \) is an integer coefficient, show that the set of all algebraic numbers is countable.

1.1.23. Show that two planes with finitely many points removed are equivalent.

1.1.24. Give another proof of Theorem 1.1.7 by checking that the map
\[ a \rightarrow \{a, -a\} \] is injective.

1.1.25. Show that the intervals \( [a, b] \) for each binary sequence \( a \) are equivalent.

1.1.26. Show that two planes with finitely many points removed are equivalent.

1.1.29. Show that the intervals \( [a, b] \) for each binary sequence \( a \) are equivalent.

1.1.30. Show that a countable union of continuum sets is a continuum set.

1.1.31. This result is used in 1.1.9. Show that any real number could be written in base \( d \) with any \( d \in \mathbb{Z}, d \geq 2 \). However two forms in base \( d \) could represent the same real number, as seen in 1.1.9. This happens only if starting from certain digits, all digits of one form are 0 and all digits of the other form are \( d - 1 \).

1.1.32. We prove that \( 2^\aleph_0 = \mathbb{C} \). We prove that \( 2^\aleph_0 \) is equivalent to \( \mathbb{R} \).

1.1.33. Show that \( 2^\aleph_0 \) is equivalent to the set of all sequences of binary digits.

1.1.34. Using 1.1.31 deduce that \(|[0, 1]| \leq |2^\aleph_0|\).

1.1.35. Consider a map \( f : 2^\aleph_0 \rightarrow [0, 2] \). For each binary sequence \( a = a_1a_2a_3 \cdots \) define \( f(a) \) as follows. If starting from a certain digit, all digits are 1, then let \( f(a) = 1.1a_2a_3 \cdots \). Otherwise let \( f(a) = 0.a_1a_2a_3 \cdots \). Show that \( f \) is injective.

Deduce that \(|2^\aleph_0| \leq |[0, 2]|\).
1.1.33 ($\mathbb{R}^2$ is equivalent to $\mathbb{R}$). * Here we prove that $\mathbb{R}^2$ is equivalent to $\mathbb{R}$, in other words, a plane is equivalent to a line. As a corollary, $\mathbb{R}^n$ is equivalent to $\mathbb{R}$.

(1) First method: Construct a map from $[0, 1) \times [0, 1)$ to $[0, 1)$ as follows. In view of 1.1.31, we only allow decimal presentations in which not all digits are 9 starting from a certain digit. The pair of two real numbers $0.a_1a_2\ldots$ and $0.b_1b_2\ldots$ corresponds to the real number $0.a_1b_1a_2b_2\ldots$. Check that this map is injective.

(2) Second method: Construct a map from $2^\mathbb{N} \times 2^\mathbb{N}$ to $2^\mathbb{N}$ as follows. The pair of two binary sequences $a_1a_2\ldots$ and $b_1b_2\ldots$ corresponds to the binary sequence $a_1b_1a_2b_2\ldots$. Check that this map is injective. Then use 1.1.32.

Note: In fact for all infinite cardinal $\omega$ we have $\omega^2 = \omega$, see [Dug66 p. 52], [Lan93 p. 888].

1.1.34 (Transfinite induction principle). An ordered set $S$ is well-ordered (được sắp tốt) if every non-empty subset $A$ of $S$ has a smallest element, i.e. $\exists a \in A$, $\forall b \in A$, $a \leq b$.

For example with the usual order, $\mathbb{N}$ is well-ordered while $\mathbb{R}$ is not.

Ernst Zermelo proved in 1904 that any set can be well-ordered, based on the Axiom of choice.

The following is a generalization of the Principle of induction.

Let $A$ be a well-ordered set. Let $P(a)$ be a statement whose truth depends on $a \in A$. If

(1) $P(a)$ is true when $a$ is the smallest element of $A$

(2) if $P(a)$ is true for all $a < b$ then $P(b)$ is true

then $P(a)$ is true for all $a \in A$.

Hint: Proof by contradiction.
1.2. Topological Space

Previously, in $\mathbb{R}^n$ or in metric spaces, we know that a function is continuous if and only if the inverse image of an open set is an open set. So in order to discuss continuity of a function from one set to another set, on each set a notion of “open subset” needs to be defined. Briefly, a topology is a system of open sets.

Definition 1.2.1. A topology on a set $X$ is a collection $\tau$ of subsets of $X$ satisfying:

1. The sets $\emptyset$ and $X$ are elements of $\tau$.
2. A union of elements of $\tau$ is an element of $\tau$.
3. A finite intersection of elements of $\tau$ is an element of $\tau$.

Elements of $\tau$ are called open sets of $X$ in the topology $\tau$.

Briefly speaking, a topology on a set $X$ is a collection of subsets of $X$ which includes $\emptyset$ and $X$ and is closed under unions and finite intersections.

A complement of an open set is called a closed set.

A set $X$ together with a topology $\tau$ is called a topological space, denoted by $(X, \tau)$ or $X$ alone if we do not need to specify the topology. An element of $X$ is often called a point.

Example 1.2.2. On any set $X$ there is the trivial topology (tôpô hiển nhiên) $\{\emptyset, X\}$.

On any set $X$ there is the discrete topology (tôpô rời rạc) whereas any point constitutes an open set. That means any subset of $X$ is open, so the topology is $2^X$.

Thus on a set there can be many topologies.

Example 1.2.3. Let $X = \{1, 2, 3\}$. The following are topologies on $X$:

1. $\tau_1 = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$.
2. $\tau_2 = \{\emptyset, \{1, 2\}, \{2, 3\}, \{2\}, \{1, 2, 3\}\}$.

The following result is useful in practice:

Proposition 1.2.4. The statement “intersection of finitely many open sets is open” is equivalent to the statement “intersection of two open sets is open”.

Example 1.2.5 (Metric space). Metric spaces are important examples of topological spaces.

Recall that, briefly, a metric space is a set equipped with a distance between every two points. Namely, a metric space is a set $X$ with a map $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

1. $d(x, y) \geq 0$ (distance is non-negative),
2. $d(x, y) = 0 \iff x = y$ (distance is zero if and only if the two points coincide),
3. $d(x, y) = d(y, x)$ (distance is symmetric),
4. $d(x, y) + d(y, z) \geq d(x, z)$ (triangular inequality).

A ball is a set of the form $B(x, r) = \{y \in X \mid d(y, x) < r\}$ where $r \in \mathbb{R}, r > 0$. 
In the theory of metric spaces, a subset \( U \) of \( X \) is said to be open if for all \( x \) in \( U \) there is \( \epsilon > 0 \) such that \( B(x, \epsilon) \) is contained in \( U \). This is equivalent to saying that a non-empty open set is a union of balls.

This is indeed a topology. We only need to check that the intersection of two balls is open. This in turn can be proved using the triangle inequality of the metric.

Let \( z \in B(x, r_x) \cap B(y, r_y) \). Let \( r_z = \min\{r_x - d(z, x), r_y - d(z, y)\} \). Then the ball \( B(z, r_z) \) will be inside both \( B(x, r_x) \) and \( B(y, r_y) \).

Thus a metric space is canonically a topological space with the topology generated by the metric. When we speak about topology on a metric space we mean this topology.

**Example 1.2.6 (Normed spaces).** Recall that a normed space (khoảng gian định chuẩn) is briefly a vector spaces equipped with lengths of vectors. Namely, a normed space is a set \( X \) with a structure of vector space over the real numbers and a real function \( x \mapsto ||x|| \), called a norm (chuẩn), satisfying:

1. \( ||x|| \geq 0 \) and \( ||x|| = 0 \iff x = 0 \) (length is non-negative),
2. \( ||cx|| = |c|||x|| \) for \( c \in \mathbb{R} \) (length is proportionate to vector),
3. \( ||x + y|| \leq ||x|| + ||y|| \) (triangle inequality).

A normed space is canonically a metric space with metric \( d(x, y) = ||x - y|| \). Therefore a normed space is canonically a topological space with the topology generated by the norm.

**Example 1.2.7 (Euclidean topology).** In \( \mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \} \), the Euclidean distance between two points \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) is \( d(x, y) = \left[ \sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2} \).

The topology generated by this metric is called the Euclidean topology (tôpô Euclid) of \( \mathbb{R}^n \).

**Example 1.2.8 (Finite complement topology).** The finite complement topology on \( X \) consists of the empty set and all subsets of \( X \) whose complements are finite.

**Proposition 1.2.9 (Dual description of topology).** In a topological space \( X \):

1. \( \emptyset \) and \( X \) are closed.
2. A finite union of closed sets is closed.
3. An intersection of closed sets is closed.

The above statement gives a dual (đối ngẫu) description of topology in term of closed set instead of open set.

A neighborhood (lân cận) of a point \( x \in X \) is a subset of \( X \) which contains an open set containing \( x \). Note that a neighborhood does not need to be open.\footnote{Be careful that not everyone uses this convention. For instance Kelley \cite{Kel55} uses this convention but Munkres \cite{Mun00} requires a neighborhood to be open.}
Bases of a topology.

Definition 1.2.10. Given a topology, a collection of open sets is a basis (cơ sở) for that topology if every non-empty open set is a union of members of that collection.

More concisely, let \( \tau \) be a topology of \( X \), then a collection \( B \subseteq \tau \) is called a basis for \( \tau \) if any non-empty member of \( \tau \) is a union of members of \( B \).

So a basis of a topology is a subset of the topology that generates the entire topology via unions. Thus specifying a basis is a more “economical way” to describe a topology.

Example 1.2.11. In a metric space the collection of all balls is a basis for the topology.

Example 1.2.12. On \( \mathbb{R}^2 \) the collection of all interiors of circles, the collection of all interiors of squares, and collection of all interiors of ellipses are bases of the Euclidean topology.

So, note that a topology may have many bases.

Definition 1.2.13. A collection \( S \subseteq \tau \) is called a subbasis (tiền cơ sở) for the topology \( \tau \) if the collection of finite intersections of members of \( S \) is a basis for \( \tau \).

Clearly a basis for a topology is also a subbasis for that topology.

Briefly, given a topology, a subbasis is a subset of the topology that can generate the entire topology by unions and finite intersections.

Example 1.2.14. Let \( X = \{1, 2, 3\} \). The topology \( \tau = \{\emptyset, \{1, 2\}, \{2, 3\}, \{2\}, \{1, 2, 3\}\} \) has a basis \( \{\{1, 2\}, \{2, 3\}\{2\}\} \) and a subbasis \( \{\{1, 2\}, \{2, 3\}\} \).

Example 1.2.15. The collection of all open rays, that are, sets of the forms \((a, \infty)\) and \((-\infty, a)\), is a subbasis for the Euclidean topology of \( \mathbb{R} \).

The collection of all open half-planes (meaning not consisting the separating lines) is a subbasis for the Euclidean topology of \( \mathbb{R}^2 \).

Comparing topologies.

Definition 1.2.16. Let \( \tau_1 \) and \( \tau_2 \) be two topologies on \( X \). If \( \tau_1 \subseteq \tau_2 \) we say that \( \tau_2 \) is finer (mịn hơn) (or stronger, bigger) than \( \tau_1 \) and \( \tau_1 \) is coarser (thô hơn) (or weaker, smaller) than \( \tau_2 \).

Example 1.2.17. On a set the trivial topology is the coarsest topology and the discrete topology is the finest one.

Generating topologies. Suppose that we have a set and we want certain subsets of that set to be open, how do find a topology for that purpose?

Theorem 1.2.18. Let \( S \) be a collection of subsets of \( X \). Then the collection \( \tau \) consisting of the empty set, \( X \), and all unions of finite intersections of members of \( S \) is the coarsest
topology on $X$ that contains $S$, called the topology generated by $S$. The collection $S \cup \{X\}$ is a subbasis for this topology.  

**Proof.** To check that $\tau$ is a topology we only need to check that the intersection of two unions of finite intersections of members of $S$ is a union of finite intersections of members of $S$.

Let $A$ and $B$ be two collections of finite intersections of members of $S$. We have $(\bigcup_{C \in A} C) \cap (\bigcup_{D \in B} D) = \bigcup_{C \in A, D \in B} (C \cap D)$. Since each $C \cap D$ is a finite intersection of elements of $S$ we get the desired conclusion. \qed

By this theorem, given a set, any collection of subsets generates a topology on that set.

Given a collection of subsets of a set, when is it a basis for a topology?

**Proposition 1.2.19.** Let $B$ be a collection of subsets of $X$. Then $B \cup \{X\}$ is a basis for a topology on $X$ if and only if the intersection of two members of $B$ is either empty or is a union of some members of $B$. If $B$ satisfies this condition, then the collection $\tau$ consists of the empty set, $X$ and unions of members of $B$ is a topology on $X$.

**Proof.** Verifying that $\tau$ is a topology is reduced to checking that the intersection of two members of $\tau$ is a member of $\tau$. Indeed, if $A$ and $C$ are two collections of members of $B$, then $(\bigcup_{D \in A} D) \cap (\bigcup_{E \in C} E) = \bigcup_{D \in A, E \in C} (D \cap E)$. By assumption $D \cap E$ is either empty or is a union of some elements of $B$. \qed

**Example 1.2.20.** Let $X = \{1, 2, 3, 4\}$. The set $\{\{1\}, \{2, 3\}, \{3, 4\}\}$ generates the topology $\{\emptyset, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. A basis for this topology is $\{\{1\}, \{3\}, \{2, 3\}, \{3, 4\}\}$.

**Example 1.2.21 (Ordering topology).** Let $(X, \leq)$ be a totally ordered set with at least two elements. The collection of subsets of the forms $\{\beta \in X \mid \beta < \alpha\}$ and $\{\beta \in X \mid \beta > \alpha\}$ generates a topology on $X$, called the *ordering topology*.

**Example 1.2.22.** The Euclidean topology on $\mathbb{R}$ is the ordering topology with respect to the usual order of real numbers. (This is just a different way to state 1.2.15.)

**Problems.**

1.2.23. A collection $B$ of open sets is a basis if for each point $x$ and each open set $O$ containing $x$ there is a $U$ in $B$ such that $U$ contains $x$ and $U$ is contained in $O$.

1.2.24. Show that two bases generate the same topology if and only if each member of one basis is a union of members of the other basis.

1.2.25. In a metric space the set of all balls with rational radii is a basis for the topology.

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6In some textbooks to avoid adding the element $X$ to $S$ it is required that the union of all members of $S$ is $X$.

7In some textbooks to avoid adding the element $X$ to $B$ it is required that the union of all members of $B$ is $X$. 

1.2.26. In a metric space the set of all balls with radii \( \frac{1}{m} \), \( m \geq 1 \) is a basis.

1.2.27. \( \mathbb{R}^n \) has a countable basis. * The set of all balls each with rational radius whose center has rational coordinates forms a basis for the Euclidean topology of \( \mathbb{R}^n \).

1.2.28. On \( \mathbb{R}^n \) consider the following norms. Let \( x = (x_1, x_2, \ldots, x_n) \).

   1. \( \|x\|_1 = \sum_{i=1}^{n} |x_i| \).
   2. \( \|x\|_2 = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2} \).
   3. \( \|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \).

Draw the unit ball for each norm. Show that these norms generate the same topologies.

1.2.29. Let \( d_1 \) and \( d_2 \) be two metrics on \( X \). If there are \( \alpha, \beta > 0 \) such that for all \( x, y \in X \),\n
   \[ \alpha d_1(x, y) < d_2(x, y) < \beta d_1(x, y) \]

then the two metrics are said to be equivalent. Show that two equivalent metrics generate the same topology.

   Hint: Show that a ball in one metric contains a ball in the other metric with the same center.

1.2.30. (All norms in \( \mathbb{R}^n \) generate the Euclidean topology). In \( \mathbb{R}^n \) denote by \( \|\cdot\|_2 \) the Euclidean norm, and let \( \|\cdot\| \) be any norm.

   1. Check that the map \( x \mapsto \|x\| \) from \( (\mathbb{R}^n, \|\cdot\|_2) \) to \( \mathbb{R} \) is continuous.
   2. Let \( S^n \) be the unit sphere under the Euclidean norm. Show that the restriction of the map above to \( S^n \) has a maximum value \( \beta \) and a minimum value \( \alpha \). Hence \( \alpha \leq \| \frac{x}{\|x\|_2} \| \leq \beta \) for all \( x \neq 0 \).

Deduce that any two norms in \( \mathbb{R}^n \) are equivalent, hence all norms in \( \mathbb{R}^n \) generate the Euclidean topology.

1.2.31. Is the Euclidean topology on \( \mathbb{R}^2 \) the same as the ordering topology on \( \mathbb{R}^2 \) with respect to the dictionary order? If it is not the same, can the two be compared?

1.2.32. Show that an open set in \( \mathbb{R} \) is a countable union of open intervals.

1.2.33. The collection of intervals of the form \([a, b)\) generates a topology on \( \mathbb{R} \). Is it the Euclidean topology?

1.2.34. Describe all sets that has only one topology.

1.2.35. * On the set of all integer numbers \( \mathbb{Z} \), consider all arithmetic progressions \( S_{a,b} = a + b\mathbb{Z} \),

where \( a \in \mathbb{Z} \) and \( b \in \mathbb{Z}^+ \).

   1. Show that these sets form a basis for a topology on \( \mathbb{Z} \).
   2. Show that with this topology each set \( S_{a,b} \) is both open and closed.
   3. Show that the set \( \{ \pm 1 \} \) is closed.
   4. Show that if there are only finitely many prime numbers then the set \( \{ \pm 1 \} \) is open.
   5. Conclude that there are infinitely many prime numbers.
1.3. Continuity

Continuous function.

Definition 1.3.1. Let $X$ and $Y$ be topological spaces. We say a map $f : X \to Y$ is continuous at a point $x$ in $X$ if for any open set $U$ of $Y$ containing $f(x)$ there is an open set $V$ of $X$ containing $x$ such that $f(V)$ is contained in $U$.

We say that $f$ is continuous on $X$ if it is continuous at every point in $X$.

Remark 1.3.2. Equivalently, $f$ is continuous at $x$ if for any open set $U$ containing $f(x)$, the set $f^{-1}(U)$ is a neighborhood of $x$. Note that we cannot require $f^{-1}(U)$ to be open.

Theorem 1.3.3. A map is continuous if and only if the inverse image of an open set is an open set.

Proof. ($\Rightarrow$) Suppose that $f : X \to Y$ is continuous. Let $U$ be an open set in $Y$. Let $x \in f^{-1}(U)$. Since $f$ is continuous at $x$ and $U$ is an open neighborhood of $f(x)$, there is an open set $V$ containing $x$ such that $V$ is contained in $f^{-1}(U)$. Thus $x$ is an interior point of $f^{-1}(U)$. Therefore $f^{-1}(U)$ is open.

($\Leftarrow$) Suppose that the inverse image of any open set is an open set. Let $x \in X$. Let $U$ be an open neighborhood of $f(x)$. Then $V = f^{-1}(U)$ is an open set containing $x$, and $f(V)$ is contained in $U$. Therefore $f$ is continuous at $x$. □

Example 1.3.4. Let $X$ and $Y$ be topological spaces.
(a) The identity function, $id_X : X \to X$, $x \mapsto x$, is continuous.
(b) The constant function, with given $a \in Y$, $x \mapsto a$, is continuous.

Proposition 1.3.5. A map is continuous if and only if the inverse image of a closed set is a closed set.

Example 1.3.6 (Metric space). Let $(X, d_1)$ and $(Y, d_2)$ be metric spaces. Recall that in the theory of metric spaces, a map $f : (X, d_1) \to (Y, d_2)$ is continuous at $x \in X$ if and only if

$$\forall \epsilon > 0, \exists \delta > 0, d_1(y, x) < \delta \Rightarrow d_2(f(y), f(x)) < \epsilon.$$ 

In other words, given any ball $B(f(x), \epsilon)$ centered at $f(x)$, there is a ball $B(x, \delta)$ centered at $x$ such that $f$ brings $B(x, \delta)$ into $B(f(x), \epsilon)$.

It is apparent that this definition is equivalent to the definition of continuity in topological spaces where the topologies are generated by the metrics.

In other words, if we look at a metric space as a topological space then continuity in the metric space is the same as continuity in the topological space. Therefore we inherit all results concerning continuity in metric spaces.

Homeomorphism. A map from one topological space to another is said to be a homeomorphism (phép đồng phôi) if it is a bijection, is continuous and its inverse map is also continuous.
Two spaces $X$ and $Y$ are said to be **homeomorphic** (đồng phôi), sometimes written $X \approx Y$, if there is a homeomorphism from one to the other.

**Example 1.3.7.** Any two open intervals in the real number line $\mathbb{R}$ under the Euclidean topology are homeomorphic.

**Proposition 1.3.8.** If $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ is a homeomorphism then it induces a bijection between $\tau_X$ and $\tau_Y$.

**Proof.** The map

$$f: \tau_X \rightarrow \tau_Y$$

$$O \mapsto f(O)$$

is a bijection. □

Roughly speaking, in the field of Topology, when two spaces are homeomorphic they are the same.

**Topology generated by maps.** Let $(X, \tau_X)$ be a topological space, $Y$ be a set, and $f : X \rightarrow Y$ be a map. We want to find a topology on $Y$ such that $f$ is continuous.

The requirement for such a topology $\tau_Y$ is that if $U \in \tau_Y$ then $f^{-1}(U) \in \tau_X$.

The trivial topology on $Y$ satisfies that requirement. It is the coarsest topology satisfying that requirement.

On the other hand the collection $$\{U \subset Y \mid f^{-1}(U) \in \tau_X\}$$ is actually a topology on $Y$. This is the finest topology satisfying that requirement.

In another situation, let $X$ be a set, $(Y, \tau_Y)$ be a topological space, and $f : X \rightarrow Y$ be a map. We want to find a topology on $X$ such that $f$ is continuous.

The requirement for such a topology $\tau_X$ is that if $U \in \tau_Y$ then $f^{-1}(U) \in \tau_X$.

The discrete topology on $X$ is the finest topology satisfying that requirement. The collection $\tau_X = \{f^{-1}(U) \mid U \in \tau_Y\}$ is the coarsest topology satisfying that requirement. We can observe further that if the collection $S_Y$ generates $\tau_Y$ then $\tau_X$ is generated by the collection $\{f^{-1}(U) \mid U \in S_Y\}$.

1.3.9. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous then $g \circ f$ is continuous.

1.3.10. * Suppose that $f : X \rightarrow Y$ and $S$ is a subbasis for the topology of $Y$. Show that $f$ is continuous if and only if the inverse image of any element of $S$ is an open set in $X$.

1.3.11. Define an open map to be a map such that the image of an open set is an open set. A closed map is a map such that the image of a closed set is a closed set.

Show that a homeomorphism is both an open map and a closed map.

1.3.12. A continuous bijection is a homeomorphism if and only if it is an open map.

1.3.13. * Let $X$ be a set and $(Y, \tau)$ be a topological space. Let $f_i : X \rightarrow Y$, $i \in I$ be a collection of maps. Find the coarsest topology on $X$ such that all maps $f_i$, $i \in I$ are continuous.

**Note:** In Functional Analysis this construction is used to construct the weak topology on a normed space. It is the coarsest topology such that all linear functionals which are continuous under the norm are still continuous under the topology. See for instance [Con90].
1.3.14. * Suppose that $X$ is a normed space. Prove that the topology generated by the norm is exactly the coarsest topology on $X$ such that the norm and the translations (maps of the form $x \mapsto x + a$) are continuous.
1.4. Subspace

Subspace topology. Let \((X, \tau)\) be a topological space and let \(A\) be a subset of \(X\). The sub\(\text{space topology}\) on \(A\), also called the relative topology, is defined to be the collection \(\{A \cap O \mid O \in \tau\}\). With this topology we say that \(A\) is a subspace of \(X\).

Thus a subset of a subspace \(A\) of \(X\) is open in \(A\) if and only if it is a restriction of a open set in \(X\) to \(A\).

**Proposition 1.4.1.** A subset of a subspace \(A\) of \(X\) is closed in \(A\) if and only if it is a restriction of a closed set in \(X\) to \(A\).

**Remark 1.4.2.** An open or a closed subset of a subspace \(A\) of a space \(X\) is not necessarily open or closed in \(X\). For example, under the Euclidean topology of \(\mathbb{R}\), the set \([0, 1/2)\) is open in the subspace \([0, 1]\), but is not open in \(\mathbb{R}\).

When we say that a set is open, we must know which space we are talking about.

**Example 1.4.3 (Stereographic projection).** Define the sphere \(S^n\) to be the subspace of the Euclidean space \(\mathbb{R}^{n+1}\) \(\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1 \}\).

Then \(S^n \setminus \{(0,0,\ldots,0,1)\}\) is homeomorphic to \(\mathbb{R}^n\) via the stereographic projection (phép chiếu nổi).

![Figure 1.4.1. The stereographic projection.](image)

The formula for this projection is:

\[
S^n \setminus \{(0,0,\ldots,0,1)\} \rightarrow \mathbb{R}^n \times \{0\}
\]

\[(x_1, x_2, \ldots, x_{n+1}) \mapsto (y_1, y_2, \ldots, y_n, 0)\]

where \(y_i = \frac{1}{1-x_{n+1}} x_i\).

**Proposition 1.4.4.** Suppose that \(X\) is a topological space and \(Z \subset Y \subset X\). Then the relative topology of \(Z\) with respect to \(Y\) is the same as the relative topology of \(Z\) with respect to \(X\).

**Embedding.** An embedding (or imbedding) (phép nhúng) from the topological space \(X\) to the topological space \(Y\) is a map \(f : X \rightarrow Y\) such that its restriction
$f : X \to f(X)$ is a homeomorphism. This means $f$ maps $X$ homeomorphically onto its image. If there is an imbedding from $X$ to $Y$, i.e. if $X$ is homeomorphic to a subspace of $Y$ then we say that $X$ can be embedded in $Y$.

**Example 1.4.5.** The Euclidean line $\mathbb{R}$ can be embedded in the Euclidean plane $\mathbb{R}^2$ as a line in the plane.

**Example 1.4.6.** Suppose that $f : \mathbb{R} \to \mathbb{R}^2$ is continuous under the Euclidean topology. Then $\mathbb{R}$ can be embedded into the plane as the graph of $f$.

**Interior – Closure – Boundary.** Let $X$ be a topological space and let $A$ be a subset of $X$. Let $x$ be a point in $X$.

The point $x$ is said to be an **interior point** of $A$ in $X$ if there is a neighborhood of $x$ that is contained in $A$.

The point $x$ is said to be a **contact point** (điểm dính) (or point of closure) of $A$ in $X$ if any neighborhood of $x$ contains a point of $A$.

The point $x$ is said to be a **limit point** (điểm tụ) (or cluster point, or accumulation point) of $A$ in $X$ if any neighborhood of $x$ contains a point of $A$ other than $x$.

Of course a limit point is a contact point. We can see that a contact point of $A$ which is not a point of $A$ is a limit point of $A$.

A point $x$ is said to be a **boundary point** (điểm biên) of $A$ in $X$ if every neighborhood of $x$ contains a point of $A$ and a point of the complement of $A$.

In other words, a boundary point of $A$ is a contact point of both $A$ and the complement of $A$.

The set of all interior points of $A$ is called the **interior** (phần trong) of $A$ in $X$, denoted by $\text{int}(A)$ or $\text{int} A$.

The set of all contact points of $A$ in $X$ is called the **closure** (bao đóng) of $A$ in $X$, denoted by $\text{cl}(A)$ or $\text{cl} A$.

The set of all boundary points of $A$ in $X$ is called the **boundary** (biên) of $A$ in $X$, denoted by $\partial A$.

**Example 1.4.7.** On the Euclidean line $\mathbb{R}$, consider the subspace $A = [0, 1) \cup \{2\}$. Its interior is $\text{int} A = (0, 1)$, the closure is $\text{cl} A = [0, 1] \cup \{2\}$, the boundary is $\partial A = \{0, 1, 2\}$, the set of all limit points is $[0, 1]$.

**Remark 1.4.8.** It is crucial to understand that the notions of interior, closure, boundary, contact points, and limit points of a space $A$ only make sense relative to a certain space $X$ containing $A$ as a subspace (there must be a “mother space”).

**Proposition 1.4.9.** The interior of $A$ in $X$ is the largest open subset of $X$ that is contained in $A$.

A subspace is open if all of its points are interior points.

**Proposition 1.4.10.** The closure of $A$ in $X$ is the smallest closed subset of $X$ containing $A$.

A subspace is closed if and only if it contains all of its contact points.
1.4. Problems.

1.4.11. Let $X$ be a topological space and let $A \subset X$. Then the subspace topology on $A$ is exactly the coarsest topology on $A$ such that the inclusion map $i : A \hookrightarrow X, x \mapsto x$ is continuous.

1.4.12. (a) Let $f : X \to Y$ be continuous and let $Z$ be a subspace of $X$. Then the restriction of $f$ to $Z$ is still continuous.

(b) Let $f : X \to Y$ be continuous and let $Z$ be a space containing $Y$ as a subspace. Consider $f$ as a function from $X$ to $Z$, then it is still continuous.

1.4.13. (Gluing continuous functions). * Let $X = A \cup B$ where $A$ and $B$ are both open or are both closed in $X$. Suppose $f : X \to Y$, and $f|_A$ and $f|_B$ are both continuous. Then $f$ is continuous.

Another way to phrase this is the following. Let $g : A \to Y$ and $h : B \to Y$ be continuous and $g(x) = h(x)$ on $A \cap B$. Define

$$f(x) = \begin{cases} g(x), & x \in A \\ h(x), & x \in B \end{cases}.$$ 

Then $f$ is continuous.

Is it still true if the restriction that $A$ and $B$ are both open or are both closed in $X$ is removed?

1.4.14. * Any two balls in a normed space are homeomorphic.

1.4.15. * A ball in a normed space is homeomorphic to the whole space.

*Hint:* Consider a map from the unit ball to the space, such as: $x \mapsto \frac{x}{\sqrt{1 - \|x\|^2}}$.

1.4.16. Is it true that any two balls in a metric space homeomorphic?

*Hint:* Consider a metric space with only two points.

1.4.17. In the Euclidean plane an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is homeomorphic to a circle.

1.4.18. In the Euclidean plane the upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ is homeomorphic to the plane.

1.4.19. In the Euclidean plane:

(1) A square and a circle are homeomorphic.

(2) The region bounded by a square and the region bounded by the circle are homeomorphic.

1.4.20. * If $f : X \to Y$ is a homeomorphism and $Z \subset X$ then $X \setminus Z$ and $Y \setminus f(Z)$ are homeomorphic.

1.4.21. On the Euclidean plane $\mathbb{R}^2$, show that:

(1) $\mathbb{R}^2 \setminus \{(0,0)\}$ and $\mathbb{R}^2 \setminus \{(1,1)\}$ are homeomorphic.

(2) $\mathbb{R}^2 \setminus \{(0,0), (1,1)\}$ and $\mathbb{R}^2 \setminus \{(1,0), (0,1)\}$ are homeomorphic.

1.4.22. Show that $\mathbb{N}$ and $\mathbb{Z}$ are homeomorphic under the Euclidean topology.

Further, prove that any two discrete spaces having the same cardinalities are homeomorphic.
1.4.23. Among the following spaces, which one is homeomorphic to another? \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \), each with the Euclidean topology, and \( \mathbb{R} \) with the finite complement topology.

1.4.24. Show that any homeomorphism from \( S^{n-1} \) onto \( S^{n-1} \) can be extended to a homeomorphism from the unit disk \( D^n = B'(0, 1) \) onto \( D^n \).

1.4.25. The closure of a subspace is the union of the subspace and the set of all of its limit points.

1.4.26. Show that \( \bar{\mathbb{A}} \) is the disjoint union of \( \mathring{\mathbb{A}} \) and \( \partial \mathbb{A} \).

1.4.27. Show that \( X \) is the disjoint union of \( \mathring{\mathbb{A}}, \partial \mathbb{A}, \) and \( X \setminus \mathbb{A} \).

1.4.28. The set \( \{ x \in \mathbb{Q} \mid -\sqrt{2} \leq x \leq \sqrt{2} \} \) is both closed and open in \( \mathbb{Q} \) under the Euclidean topology of \( \mathbb{R} \).

1.4.29. The map \( \phi : \left[ 0, 2\pi \right) \rightarrow S^1 \) given by \( t \mapsto (\cos t, \sin t) \) is a bijection but is not a homeomorphism, under the Euclidean topology.

* Hint: Compare the subinterval \( [1, 2\pi) \) and its image via \( \phi \).

1.4.30. Find the closures, interiors and the boundaries of the interval \( [0, 1) \) under the Euclidean, discrete and trivial topologies of \( \mathbb{R} \).

1.4.31. * In a metric space \( X \), a point \( x \in X \) is a limit point of the subset \( A \) of \( X \) if and only if there is a sequence in \( A \setminus \{ x \} \) converging to \( x \).

* Note: This is not true in general topological spaces.

1.4.32. * In a normed space, show that the boundary of the ball \( B(x, r) \) is the sphere \( \{ y \mid d(x, y) = r \} \), and so the ball \( B'(x, r) = \{ y \mid d(x, y) \leq r \} \) is the closure of \( B(x, r) \).

* Hint: Consider a metric space consisting of two points.

1.4.33. In a metric space, show that the boundary of the ball \( B(x, r) \) is a subset of the sphere \( \{ y \mid d(x, y) = r \} \). Is the closed ball \( B'(x, r) = \{ y \mid d(x, y) \leq r \} \) the closure of \( B(x, r) \)?

* Hint: Consider a metric space consisting of two points.

1.4.34. Suppose that \( A \subset Y \subset X \). Then \( \overline{A}^Y = \overline{A}^X \cap Y \). Furthermore if \( Y \) is closed in \( X \) then \( \overline{Y}^X = \overline{Y}^X \).

1.4.35. Let \( O_n = \{ k \in \mathbb{Z}^+ \mid k \geq n \} \). Check that \( \{ \emptyset \} \cup \{ O_n \mid n \in \mathbb{Z}^+ \} \) is a topology on \( \mathbb{Z}^+ \). Find the closure of the set \( \{ 5 \} \). Find the closure of the set of all even positive integers.

1.4.36. Verify the following properties.

1. \( X \setminus \mathring{A} = \overline{X} \setminus \overline{A} \).
2. \( X \setminus \overline{A} = X \setminus \mathring{A} \).
3. If \( A \subset B \) then \( \mathring{A} \subset \mathring{B} \).

1.4.37. Which ones of the following equalities are correct?

1. \( A \cup \mathring{B} = \mathring{A} \cup \mathring{B} \).
2. \( A \cap \mathring{B} = \mathring{A} \cap \mathring{B} \).
3. \( \overline{A \cup B} = \overline{A} \cup \overline{B} \).
4. \( \overline{A \cap B} = \overline{A} \cap \overline{B} \).
1.5. Connectedness

A topological space is said to be connected (liên thông) if it is not a union of two non-empty disjoint open subsets.

Equivalently, a topological space is connected if and only if its only subsets which are both closed and open are the empty set and the space itself.

Remark 1.5.1. When we say that a subset of a topological space is connected we implicitly mean that the subset under the subspace topology is a connected space.

Example 1.5.2. The Euclidean real number line minus a point is no longer connected.

Proposition 1.5.3. If two connected subspaces of a space have non-empty intersection then their union is connected.

Proof. Let \( X \) be a topological space and let \( A \) and \( B \) be two connected subspaces. Suppose that \( A \cap B \neq \emptyset \). Suppose that \( C \) is a subset of \( A \cup B \) that is both open and closed in \( A \cup B \). Suppose that \( C \neq \emptyset \). Then either \( C \cap A \neq \emptyset \) or \( C \cap B \neq \emptyset \). Without loss of generality, assume that \( C \cap A \neq \emptyset \).

Note that \( C \cap A \) is both open and closed in \( A \) (we are using 1.4.4 here). Since \( A \) is connected, \( C \cap A = A \). Then \( C \cap B \neq \emptyset \), so the same argument shows that \( C \cap B = B \), hence \( C = A \cup B \). \( \square \)

The same proof gives a more general result:

Proposition 1.5.4. If a collection of connected subspaces of a space have non-empty intersection then their union is connected.

Proposition 1.5.5 (Continuous image of connected space is connected). If \( f : X \to Y \) is continuous and \( X \) is connected then \( f(X) \) is connected.

Connected component. Let \( X \) be a topological space. Define a relation on \( X \) whereas two points are related if both belong to a connected subspace of \( X \) (we say that the two points are connected). Then this relation is an equivalence relation, by 1.5.3.

Proposition 1.5.6. An equivalence class under the above equivalence relation is connected.

Proof. Consider the equivalence class \([a]\) represented by a point \(a\). By the definition, \(b \in [a]\) if and only if there is a connected set \(O_b\) containing both \(a\) and \(b\). Thus \([b] = \bigcup_{b \in [a]} O_b\). By 1.5.4 \([a]\) is connected. \( \square \)

Definition 1.5.7. Under the above equivalence relation, the equivalence classes are called the connected components of the space.

Thus a space is a disjoint union of its connected components.
Proposition 1.5.8. A connected component is a maximal connected subset under the set inclusion.

Theorem 1.5.9. If two spaces are homeomorphic then they have the same number of connected components, i.e. there is a bijection between the collections of connected components of the two spaces.

In particular, if two spaces are homeomorphic and one space is connected then the other space is also connected.

We say that the number of connected components is a topological invariant.

Proposition 1.5.10. A subspace with a limit point added is still connected. As a corollary, the closure of a connected subspace is connected.

Proof. Let $A$ be a connected subspace of a space $X$ and let $a \notin A$ be a limit point of $A$, we show that $A \cup \{a\}$ is connected.

Let $B$ be a non-empty, open and closed subset of $A \cup \{a\}$. Then $B \cap A$ is both open and closed in $A$, by 1.4.4. If $B \cap A = \emptyset$ then $B = \{a\}$. There is an open subset $O$ of $X$ such that $B = O \cap (A \cup \{a\})$. That implies that $O$ contains $a$ but does not contain any point of $A$, contradicting the assumption that $a$ is a limit point of $A$.

So $B \cap A = A$. Thus $A \subset B$. There is a closed subset $C$ of $X$ such that $B = C \cap (A \cup \{a\})$. Since $a$ is a limit point of $A$ in $X$ and $A \subset C$ we must have $a \in C$. This implies $a \in B$, so $B = A \cup \{a\}$. □

Corollary 1.5.11. A connected component must be closed.

Connected sets in the Euclidean real number line.

Proposition 1.5.12. A connected subspace of the Euclidean real number line must be an interval.

Proof. Suppose that a subset $A$ of $\mathbb{R}$ is connected. Suppose that $x, y \in A$ and $x < y$. If $x < z < y$ we must have $z \in A$, otherwise the set $\{a \in A \mid a < z\} = \{a \in A \mid a \leq z\}$ will be both closed and open in $A$. Thus $A$ contains the interval $[x, y]$.

Let $a = \inf A$ if $A$ is bounded from below and $a = -\infty$ otherwise. Similarly let $b = \sup A$ if $A$ is bounded from above and $b = \infty$ otherwise. Suppose that $A$ contains more than one element. There are sequences $\{a_n\}_{n \in \mathbb{Z}^+}$ and $\{b_n\}_{n \in \mathbb{Z}^+}$ of elements in $A$ such that $a < a_n < b_n < b$ and $a_n \to a$ while $b_n \to b$. By the above argument, $[a_n, b_n] \subset A$ for all $n$. So $(a, b) \subset \bigcup_{n=1}^{\infty} [a_n, b_n] \subset A$. It follows that $A$ is either $(a, b)$ or $[a, b)$ or $(a, b]$ or $[a, b]$. □

Proposition 1.5.13. The Euclidean real number line is connected.

Proof. In this proof we use the property that the set of all real numbers is complete.
Let $C$ be a non-empty proper, both open and closed subset of $\mathbb{R}$.
Let $x \notin C$. The set $D = C \cap (-\infty, x)$ is the same as the set $C \cap (-\infty, x]$. This implies that $D$ is both open and closed in $\mathbb{R}$, and is bounded from above.

If $D \neq \emptyset$, let $s = \text{sup } D$. Since $D$ is closed and $s$ is a contact point of $D$, $s \in D$.

Since $D$ is open $s$ must belong to an open interval contained in $D$. But then there are points in $D$ which are bigger than $s$, a contradiction.

If $D = \emptyset$ we let $E = C \cap (x, \infty)$, consider $t = \text{inf } E$ and proceed similarly. □

**Theorem 1.5.14.** A subset of the Euclidean real number line is connected if and only if it is an interval.

**Proof.** By homeomorphisms we just need to consider the intervals $(0, 1)$, $(0, 1]$, and $[0, 1]$. Note that $[0, 1]$ is the closure of $(0, 1)$, and $(0, 1] = (0, 3/4) \cup [1/2, 1]$.

Or we can modify the proof of 1.5.13. □

**Example 1.5.15.** The Euclidean line $\mathbb{R}$ is itself connected. Since the Euclidean $\mathbb{R}^n$ is the union of all lines passing through the origin, it is connected.

**Path-connected Space.** Let $X$ be a topological space and let $a$ and $b$ be two points of $X$. A path (đường đi) in $X$ from $x$ to $y$ is a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$, where the interval $[a, b]$ has the Euclidean topology.

The space $X$ is said to be path-connected (liên thông đường) if for any two different points $x$ and $y$ in $X$ there is a path in $X$ from $x$ to $y$.

**Example 1.5.16.** A normed space is path-connected, and so is any convex subspace of that space: any two points $x$ and $y$ are connected by a straight line segment $x + t(y - x)$, $t \in [0, 1]$.

**Example 1.5.17.** In a normed space, the sphere $S = \{x \mid ||x|| = 1\}$ is path-connected. One way to show this is as follows. If two points $x$ and $y$ are not opposite then they can be connected by the arc $\frac{x + t(y - x)}{||x + t(y - x)||}$, $t \in [0, 1]$. If $x$ and $y$ are opposite, we can take a third point $z$, then compose a path from $x$ to $z$ with a path from $z$ to $y$.

**Proposition 1.5.18.** Let $X$ be a topological space. Define a relation on $X$ whereas a point $x$ is related to a point $y$ there is a path in $X$ from $x$ to $y$. Then this is an equivalence relation.

An equivalence class under the above equivalence relation is called a path-connected component.

**Proposition 1.5.19.** A path-connected component is a maximal path-connected subset under the set inclusions.

**Theorem 1.5.20.** A path-connected space is connected.

**Proof.** This is a consequence of the fact that an interval on the Euclidean real number line is connected. Let $X$ be path-connected. Let $x, y \in X$. There is a
path from \( x \) to \( y \). The image of this path is a connected subspace of \( X \). That means every point \( y \) belongs to the connected component containing \( x \). Therefore \( X \) has only one connected component.

A topological space is said to be \textit{locally path-connected} if every neighborhood of a point contains a path-connected neighborhood of that point.

\textbf{Example 1.5.21.} All open sets in a normed space are locally path-connected.

Generally, the reverse statement of 1.5.20 is not correct. However we have:

\textbf{Proposition 1.5.22.} A connected, locally path-connected space is path-connected.

\textbf{Proof.} Suppose that \( X \) is connected and locally path-connected. Let \( C \) be a path-connected component of \( X \). If \( x \in X \) is a contact point of \( C \) then there is a path-connected neighborhood \( U \) in \( X \) of \( x \) such that \( U \cap C \neq \emptyset \). By 1.5.40, \( U \cup C \) is path-connected, we have \( U \subset C \). This implies that \( C \) is both open and closed in \( X \), hence \( C = X \). \( \square \)

\textbf{Topologist’s sine curve.} The closure in the Euclidean plane of the graph of the function \( y = \sin \frac{1}{x}, x > 0 \) is often called the \textit{Topologist’s sine curve}. This is a classic example of a space which is connected but is not path-connected.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{topologist_sine_curve.png}
\caption{Topologist’s sine curve.}
\end{figure}

Denote \( A = \{(x, \sin \frac{1}{x}) \mid x > 0\} \) and \( B = \{0\} \times [-1, 1] \). Then the Topologist’s sine curve is \( X = A \cup B \).

\textbf{Proposition 1.5.23.} The Topologist’s sine curve is connected.

\textbf{Proof.} By 1.5.30 the set \( A \) is connected. Each point of \( B \) is a limit point of \( A \), so by 1.5.10 \( X \) is connected. \( \square \)

\textbf{Proposition 1.5.24.} The Topologist’s sine curve is not path-connected.
Proof. Suppose that there is a path \( \gamma(t) = (x(t), y(t)), t \in [0, 1] \) from the origin \((0,0)\) on \(B\) to a point on \(A\), we show that there is a contradiction.

Let \( t_0 = \sup \{t \in [0, 1] \mid x(t) = 0\} \). Then \( x(t_0) = 0, t_0 < 1\), and for all \( t > t_0 \) we have \( x(t) > 0\). Thus \( t_0 \) is the moment when the path \( \gamma \) departs from \( B \). We can see that the path jumps immediately when it departs from \( B \). Thus we will show that \( \gamma(t) \) cannot be continuous at \( t_0 \), by showing that for any \( \delta > 0 \) there are \( t_1, t_2 \in (t_0, t_0 + \delta) \) such that \( y(t_1) = 1 \) and \( y(t_2) = -1 \).

To find \( t_1 \), note that the set \( x([t_1, t_0 + \frac{\delta}{2}]) \) is an interval \([0, x_0]\) where \( x_0 > 0 \). There exists an \( x_1 \in (0, x_0) \) such that \( \sin \frac{1}{x_1} = 1 \): we just need to take \( x_1 = \frac{1}{\pi + 2\pi n} \) with sufficiently large \( n \). There is \( t_1 \in (t_0, t_0 + \frac{\delta}{2}) \) such that \( x(t_1) = x_1 \). Then \( y(t_1) = \sin \frac{1}{x_1} = 1 \). We can find \( t_2 \) similarly. \(\square\)

The Borsuk-Ulam theorem. Below is a simple version of the Borsuk-Ulam theorem:

**Theorem 1.5.25 (Borsuk-Ulam theorem).** If a real function on a sphere \( S^n \) is continuous then there must be two antipodal points on the sphere where the values of the function are same. \(\textsuperscript{[8]}\)

Proof. Let \( f : S^n \to \mathbb{R} \) be continuous. Let \( g(x) = f(x) - f(-x) \). Then \( g \) is continuous and \( g(x) = -g(-x) \). If there is an \( x \) such that \( g(x) \neq 0 \) then \( g(x) \) and \( g(-x) \) have opposite signs. Since \( S^n \) is connected (see \( 1.5.28 \)), the range \( g(S^n) \) is a connected subset of the Euclidean \( \mathbb{R} \), and so is an interval, containing the interval \([g(x), g(-x)]\). So 0 is in the range of \( g \) (this is a form of Intermediate value theorem). \(\square\)

Problems.

1.5.26. A space is connected if whenever it is a union of two non-empty disjoint subsets, then one of them must contain a contact point of the other one.

1.5.27. Here is a different proof of \( 1.5.13 \) Suppose that \( A \) and \( B \) are non-empty, disjoint subsets of \((0,1)\) whose union is \((0,1)\). Let \( a \in A \) and \( b \in B \). Let \( a_0 = a, b_0 = b \), and for each \( n \geq 1 \) consider the middle point of the segment from \( a_n \) to \( b_n \). If \( \frac{a_n + b_n}{2} \in A \) then let \( a_{n+1} = \frac{a_n + b_n}{2} \) and \( b_{n+1} = b_n \); otherwise let \( a_{n+1} = a_n \) and \( b_{n+1} = \frac{a_n + b_n}{2} \). Then:

1. The sequence \( \{a_n \mid n \geq 1\} \) is a Cauchy sequence, hence is convergent to a number \( a \).

2. The sequence \( \{b_n \mid n \geq 1\} \) is also convergent to \( a \). This implies that \((0,1)\) is connected.

1.5.28. Show that the sphere \( S^n \) is connected.

1.5.29 (Intermediate value theorem). If \( X \) is a connected space and \( f : X \to \mathbb{R} \) is continuous, where \( \mathbb{R} \) has the Euclidean topology, then the image \( f(X) \) is an interval.

A consequence is the following familiar theorem in Calculus: Let \( f : [a, b] \to \mathbb{R} \) be continuous under the Euclidean topology. If \( f(a) \) and \( f(b) \) have opposite signs then the equation \( f(x) = 0 \) has a solution.

\(\textsuperscript{[8]}\) On the surface of the Earth at any moment there are two opposite places where temperatures are same!
1.5.30. * If \( f : \mathbb{R} \to \mathbb{R} \) is continuous under the Euclidean topology then its graph is connected in the Euclidean plane. Moreover the graph is homeomorphic to \( \mathbb{R} \).

1.5.31. Let \( X \) be a topological space and let \( A_i, i \in I \) be connected subspaces. If \( A_i \cap A_j \neq \emptyset \) for all \( i \neq j \) then \( \bigcup_i A_i \) is connected.

1.5.32. Let \( X \) be a topological space and let \( A_i, i \in \mathbb{Z}^+ \) be connected subsets. If \( A_i \cap A_{i+1} \neq \emptyset \) for all \( i \geq 1 \) then \( \bigcup_{i=1}^\infty A_i \) is connected.

*Hint:* Note that the conclusion is different from that \( \bigcup_{i=1}^n A_i \) is connected for all \( n \in \mathbb{Z}^+ \).

1.5.33. Is an intersection of connected subsets of a space connected?

1.5.34. Let \( X \) be connected and let \( f : X \to Y \) be continuous. If \( f \) is locally constant on \( X \) (meaning that every point has a neighborhood on which \( f \) is a constant map) then \( f \) is constant on \( X \).

1.5.35. Let \( X \) be a topological space. A map \( f : X \to Y \) is called a discrete map if \( Y \) has the discrete topology and \( f \) is continuous. Show that \( X \) is connected if and only if all discrete maps on \( X \) are constant.

Use this criterion to prove some of the results in this section.

1.5.36. What are the connected components of \( \mathbb{N} \) and \( \mathbb{Q} \) on the Euclidean real number line?

1.5.37. * What are the connected components of \( \mathbb{Q}^2 \) on the Euclidean plane?

1.5.38. Show that if a space has finitely many components then each component is both open and closed.

Is it still true if there are infinitely many components?

1.5.39. Suppose that a space \( X \) has finitely many connected components. Show that a map defined on \( X \) is continuous if and only if it is continuous on each components.

Is it still true if \( X \) has infinitely many components?

1.5.40. If a collection of path-connected subspaces of a space has non-empty intersection then its union is path-connected.

1.5.41. If \( f : X \to Y \) is continuous and \( X \) is path-connected then \( f(X) \) is path-connected.

1.5.42. If two space are homeomorphic then there is a bijection between the collections of path-connected components of the two spaces. In particular, if one space is path-connected then the other space is also path-connected.

1.5.43. The plane with a point removed is path-connected under the Euclidean topology.

1.5.44. The plane with countably many points removed is path-connected under the Euclidean topology.

*Hint:* Let \( A \) be countable and \( a \in \mathbb{R}^2 \setminus A \). There is a line \( \ell_a \) passing through \( a \) that does not intersect \( A \) (by an argument involving cardinalities of sets).

1.5.45. * The Euclidean line and the Euclidean plane are not homeomorphic.

*Hint:* Delete a point from \( \mathbb{R} \). Use 1.4.20 and 1.5.9

1.5.46. Find as many ways to prove that \( S^n \) is path-connected as you can!
1.5.47. Show that $\mathbb{R}$ with the finite complement topology and $\mathbb{R}^2$ with the finite complement topology are homeomorphic.

1.5.48. A topological space is locally path-connected if and only if the collection of all open path-connected subsets is a basis for the topology.

1.5.49. The Topologist’s sine curve is not locally path-connected.

1.5.50. * Classify the alphabetical characters up to homeomorphisms, that is, which of the following characters are homeomorphic to each other as subsets of the Euclidean plane?

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

Note that the result depends on the font you use!

Do the same for the Vietnamese alphabetical characters.

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

Hint: Use [1.4.13] to modify a letter part by part.

Further readings

Invariance of dimension. That the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic is not easy. It is a consequence of the following difficult theorem of L. Brouwer in 1912:

**Theorem 1.5.51 (Invariance of dimension).** If two subsets of the Euclidean $\mathbb{R}^n$ are homeomorphic and one set is open then the other is also open.

This theorem is often proved using Algebraic Topology, see for instance [Mun00] p. 381], [Vic94] p. 34], [Hat01] p. 126].

**Corollary 1.5.52.** The Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $m \neq n$.

**Proof.** Suppose that $m < n$. It is easy to check that the inclusion map $\mathbb{R}^m \rightarrow \mathbb{R}^n$, $(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_m, 0, \ldots, 0)$ is a homeomorphism onto its image $A \subset \mathbb{R}^n$. If $A$ is homeomorphic to $\mathbb{R}^m$ then by Invariance of dimension, $A$ is open in $\mathbb{R}^m$. But $A$ is clearly not open in $\mathbb{R}^n$. □

This result allows us to talk about topological dimension.

**Jordan curve theorem.** The following is an important and deep result of plane topology.

**Theorem 1.5.53 (Jordan curve theorem).** A simple, continuous, closed curve separates the plane into two disconnected regions. More concisely, if $C$ is a subset of the Euclidean plane homeomorphic to the circle then $\mathbb{R}^2 \setminus C$ has two connected components.

Nowadays this theorem is usually proved in a course in Algebraic Topology.

**Space filling curves.** A rather curious and surprising result is:

**Theorem 1.5.54.** There is a continuous curve filling a rectangle on the plane. More concisely, there is a continuous map from the interval $[0, 1]$ onto the square $[0, 1]^2$ under the Euclidean topology.

Note that this map cannot be injective, in other words the curve cannot be simple.

Such a curve is called a *Peano curve*. It could be constructed as a limit of an iteration of piecewise linear curves.
1.6. Separation

Definition 1.6.1. We define:

$T_1$: A topological space is called a $T_1$-space if for any two points $x \neq y$ there is an open set containing $x$ but not $y$ and an open set containing $y$ but not $x$.

$T_2$: A topological space is called a $T_2$-space or Hausdorff if for any two points $x \neq y$ there are disjoint open sets $U$ and $V$ such that $x \in U$ and $y \in V$.

$T_3$: A $T_1$-space is called a $T_3$-space or regular (chính tắc) if for any point $x$ and a closed set $F$ not containing $x$ there are disjoint open sets $U$ and $V$ such that $x \in U$ and $F \subset V$.

$T_4$: A $T_1$-space is called a $T_4$-space or normal (chuẩn tắc) if for any two disjoint closed sets $F$ and $G$ there are disjoint open sets $U$ and $V$ such that $F \subset U$ and $G \subset V$.

These definitions are often called separation axioms because they involve “separating” certain sets from each other by open sets.

Proposition 1.6.2. A space is $T_1$ if and only if a set containing exactly one point is a closed set.

Corollary 1.6.3 ($T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$). If a space is $T_i$ then it is $T_{i-1}$, for $2 \leq i \leq 4$.

Example 1.6.4. Any space with the discrete topology is normal.

Example 1.6.5. Let $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}\}$. Then $X$ is $T_0$ but is not $T_1$.

Example 1.6.6. The real number line under the finite complement topology is $T_1$ but is not $T_2$.

Remark 1.6.7. There are examples of a $T_2$-space which is not $T_3$, and a $T_3$-space which is not $T_4$, but they are rather difficult, see [1.6.16, 1.11.12] [Mun00] p. 197] and [SJ70].

Proposition 1.6.8. Any metric space is normal.

Proof. We introduce the notion of distance between two sets in a metric space $X$. If $A$ and $B$ are two subsets of $X$ then we define the distance between $A$ and $B$ as $d(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}$. In particular if $x \in X$ then $d(x, A) = \inf\{d(x, y) \mid y \in A\}$. Using the triangle inequality we can check that $d(x, A)$ is a continuous function with respect to $x$.

Now suppose that $A$ and $B$ are disjoint closed sets. Let $U = \{x \mid d(x, A) < d(x, B)\}$ and $V = \{x \mid d(x, A) > d(x, B)\}$. Then $A \subset U$, $B \subset V$, $U \cap V = \emptyset$, and both $U$ and $V$ are open.

Proposition 1.6.9. A $T_1$-space $X$ is regular if and only if given a point $x$ and an open set $U$ containing $x$ there is an open set $V$ such that $x \in V \subset \overline{V} \subset U$.

Some authors do not include the $T_1$ requirement.
PROOF. Suppose that $X$ is regular. Since $X \setminus U$ is closed and disjoint from $C$ there is an open set $V$ containing $x$ and an open set $W$ containing $X \setminus U$ such that $V$ and $W$ are disjoint. Then $V \subset (X \setminus W)$, so $V \subset (X \setminus U) \subset U$.

Now suppose that $X$ is $T_1$ and the condition is satisfied. Given a point $x$ and a closed set $C$ disjoint from $x$. Let $U = X \setminus C$. Then there is an open set $V$ containing $x$ such that $V \subset V \subset U$. Then $V$ and $X \setminus V$ separate $x$ and $C$. □

Similarly we have:

Proposition 1.6.10. A $T_1$-space $X$ is normal if and only if given a closed set $C$ and an open set $U$ containing $C$ there is an open set $V$ such that $C \subset V \subset V \subset U$.

Problems.

1.6.11. If a finite set is a $T_1$-space then the topology is the discrete topology.

1.6.12. Prove directly that any metric space is regular.

1.6.13. A subspace of a Hausdorff space is Hausdorff.

1.6.14. Let $X$ be Hausdorff and let $f : X \to Y$ be continuous. Is $f(X)$ Hausdorff?

1.6.15. A closed subspace of a normal space is normal.
   
   Note: A subspace of a normal space might not be normal.

1.6.16. Show that the set $\mathbb{R}$ with the topology generated by all the subsets of the form $(a, b)$ and $(a, b) \cap \mathbb{Q}$ is Hausdorff but is not regular.
   
   Hint: Consider the set of all irrational numbers.

1.6.17. * Let $X$ be normal, let $f : X \to Y$ be surjective, continuous, and closed. Prove that $Y$ is normal.
1.7. Convergence

In metric spaces we can study continuity of functions via convergence of sequences. In general topological spaces, we need to use a notion more general than sequences, called nets. Roughly speaking in general topological spaces, sequences (countable indexes) might not be enough to describe the neighborhood systems at a point, we need something of arbitrary index.

A directed set is a (partially) ordered set \( I \) such that for any two indices \( i \) and \( j \) there is an index \( k \) that is greater than or is equal to both \( i \) and \( j \). In symbols: \( \forall i, j \in I, \exists k \in I, k \geq i \land k \geq j \).

A net (lưới) (also called a generalized sequence) is a map from a directed set to a space. In other words, a net on a space \( X \) with index set a directed set \( I \) is a map \( x: I \rightarrow X \). It is an element of the set \( \prod_{i \in I} X \). Thus, writing \( x_i = x(i) \), we often denote the net as \( (x_i)_{i \in I} \). The notation \( \{x_i\}_{i \in I} \) is also used.

**Example 1.7.1.** Nets with index set \( I = \mathbb{N} \) with the usual order are exactly sequences.

**Example 1.7.2.** Let \( X \) be a topological space and \( x \in X \). Let \( I \) be the family of open neighborhoods of \( x \). Define an order on \( I \) by \( U \leq V \iff U \supseteq V \). Then \( I \) becomes a directed set.

It will be clear in a moment that this is the most important example of directed sets concerning nets.

**Convergence.**

**Definition 1.7.3.** A net \( (x_i) \) is said to be convergent (hội tụ) to \( x \in X \) if for each neighborhood \( U \) of \( x \) there is an index \( i \in I \) such that if \( j \geq i \) then \( x_j \) belongs to \( U \). The point \( x \) is called a limit of the net \( \{x_i\}_{i \in I} \), and we often write \( x_i \rightarrow x \).

**Example 1.7.4.** Convergence of nets with index set \( I = \mathbb{N} \) is exactly convergence of sequences.

**Example 1.7.5.** Let \( X = \{x_1, x_2, x_3\} \) with topology \( \{\emptyset, X, \{x_1, x_3\}, \{x_2, x_3\}, \{x_3\}\} \). The net \( (x_3) \) converges to \( x_1, x_2, \) and \( x_3 \). The net \( (x_1, x_2) \) converges to \( x_2 \).

If \( X \) has the trivial topology then any net in \( X \) is convergent to any point in \( X \).

**Proposition 1.7.6.** A point \( x \in X \) is a limit point of a subset \( A \subseteq X \) if and only if there is a net in \( A \setminus \{x\} \) convergent to \( x \).

This proposition allows us to describe topologies in terms of convergences. With it many statements about convergence in metric spaces could be carried to topological spaces by simply replacing sequences by nets.

**Proof.** \((\Rightarrow)\) Suppose that \( x \) is a limit point of \( A \). Consider the directed set \( I \) consisting of all the open neighborhoods of \( x \) with the partial order \( U \leq V \) if \( U \supseteq V \).
For any open neighborhood $U$ of $x$ there is an element $x_U \in U \cap A$, $x_U \neq x$. Consider the net $\{x_U\}_{U \in I}$. It is a net in $A \setminus \{x\}$ convergent to $x$. Indeed, given an open neighborhood $U$ of $x$, for all $V \supseteq U$, $x_V \in V \subset U$.

($\Leftarrow$) Suppose that there is a net $(x_i)_{i \in I}$ in $A \setminus \{x\}$ convergent to $x$. Let $U$ be an open neighborhood of $x$. There is an $i \in I$ such that for $j \geq i$ we have $x_j \in U$, in particular $x_i \in U \cap (A \setminus \{x\})$.

Remark 1.7.7 (When can nets be replaced by sequences?). By examining the above proof we can see that the term net can be replaced by the term sequence if there is a countable collection $F$ of neighborhoods of $x$ such that any neighborhood of $x$ contains a member of $F$. In other words, the point $x$ has a countable neighborhood basis. A space having this property at every point is said to be a first countable space. A metric space is such a space, where for example each point has a neighborhood basis consisting of balls of rational radii. See also 1.7.16.

Similarly to the case of metric spaces we have:

Theorem 1.7.8. Let $X$ and $Y$ be topological spaces. Then $f : X \to Y$ is continuous at $x$ if and only if whenever a net $n$ in $X$ converges to $x$, the net $f \circ n$ converges to $f(x)$.

In more familiar notations, $f$ is continuous at $x$ if and only if for all nets $(x_i)$, $x_i \to x \Rightarrow f(x_i) \to f(x)$.

Proof. The proof is simply a repeat of the proof for the case of metric spaces.

($\Rightarrow$) Suppose that $f$ is continuous at $x$. Let $U$ is a neighborhood of $f(x)$. Then $f^{-1}(U)$ is a neighborhood of $x$ in $X$. Since $(x_i)$ is convergent to $x$, there is an $i \in I$ such that for all $j \geq i$ we have $x_j \in f^{-1}(U)$, which implies $f(x_i) \in U$.

($\Leftarrow$) We will show that if $U$ is an open neighborhood in $Y$ of $f(x)$ then $f^{-1}(U)$ is a neighborhood in $X$ of $x$. Suppose the contrary, then $x$ is not an interior point of $f^{-1}(U)$, so it is a limit point of $X \setminus f^{-1}(U)$. There is a net $(x_i)$ in $X \setminus f^{-1}(U)$ convergent to $x$. Since $f$ is continuous, $f(x_i) \in Y \setminus U$ is convergent to $f(x) \in U$. That contradicts the assumption that $U$ is open.

Proposition 1.7.9. If a space is Hausdorff then a net has at most one limit.

Proof. Suppose that a net $(x_i)$ is convergent to two different points $x$ and $y$. Since the space is Hausdorff, there are disjoint open neighborhoods $U$ and $V$ of $x$ and $y$. There is $i \in I$ such that for $\gamma \geq i$ we have $x_\gamma \in U$, and there is $j \in I$ such that for $\gamma \geq j$ we have $x_\gamma \in U$. Since there is a $\gamma \in I$ such that $\gamma \geq i$ and $\gamma \geq j$, the point $x_\gamma$ will be in $U \cap V$, a contradiction.

Problems.

1.7.10. Let $I = (0, \infty)$. If $a, b \in I$, define $a \leq b$ if $a \geq b$. On the Euclidean line, define a net $x_a = a/2$ for $a \in I$. Is this net convergent?

1.7.11. Let $\mathbb{R}$ have the finite complement topology. Let $(x_i)_{i \in \mathbb{R}}$ be a net of distinct points in $\mathbb{R}$, i.e. $x_i \neq x_j$ if $i \neq j$. Where does this net converge to?
1.7.12. Suppose that $\tau_1$ and $\tau_2$ are two topologies on $X$. Show that if for all nets $x_i$ and all points $x, x_i \xrightarrow{\tau_1} x \Rightarrow x_i \xrightarrow{\tau_2} x$ then $\tau_1 \supset \tau_2$.

In other words, if convergence in $\tau_1$ implies convergence in $\tau_2$ then $\tau_1$ is finer than $\tau_2$.

Is the converse true?

*Hint:* Consider the identity map on $X$.

1.7.13. Let $X$ be a topological space, $\mathbb{R}$ have the Euclidean topology and $f : X \rightarrow \mathbb{R}$ be continuous. Suppose that $A \subset X$ and $f(x) = 0$ on $A$. Show that $f(x) = 0$ on $\overline{A}$, by:

(1) using nets.

(2) not using nets.

1.7.14. * Let $Y$ be Hausdorff and let $f, g : X \rightarrow Y$ be continuous. Show that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in $X$, by:

(1) using nets.

(2) not using nets.

Show that, as a consequence, if $f$ and $g$ agree on a dense (trù mát) subspace of $X$ (meaning the closure of that subspace is $X$) then they agree on $X$.

1.7.15. The converse statement of 1.7.9 is also true. A space is Hausdorff if and only if a net has at most one limit.

*Hint:* Suppose that there are two points $x$ and $y$ that could not be separated by open sets. Consider the directed set whose elements are pairs $(U, V)$ of open neighborhoods of $x$ and $y$, under set inclusion. Take a net $n$ such that $n(U_i, V_i)$ is a point in $U_i \cap V_i$.

1.7.16 *(Sequence is not adequate for describing convergence).* Let $(A, \leq)$ be a well-ordered uncountable set (see 1.1.34). If $A$ does not have a biggest element then add an element to $A$ and define that element to be the biggest one. Thus we can assume now that $A$ has a biggest element, denoted by $\infty$. For $a, b \in A$ denote $[a, b] = \{x \in A \mid a \leq x \leq b\}$ and $(a, b) = \{x \in A \mid a < x < b\}$. For example we can write $A = [0, \infty]$.

Let $\Omega$ be the smallest element of the set $\{a \in A \mid [0, a] \text{ is countable}\}$ (this set is non-empty since it contains $\infty$).

(1) Show that $[0, \Omega)$ is uncountable, and for all $a \in A$, $a < \Omega$ the set $[0, a]$ is countable.

(2) Show that every countable subset of $[0, \Omega)$ is bounded in $[0, \Omega]$.

(3) Consider $[0, \Omega]$ with the order topology. Show that $\Omega$ is a limit point of $[0, \Omega]$.

(4) However, show that a sequence in $[0, \Omega)$ cannot converge to $\Omega$.

So this is an example of a space for which sequence is not adequate for describing convergence.

*Hint:* (b) Let $C$ be a countable subset of $[0, \Omega)$. The set $\bigcup_{c \in C} [0, c)$ is countable while the set $[0, \Omega)$ is uncountable.

1.7.17 *(Filter).* A filter (loc) on a set $X$ is a collection $F$ of non-empty subsets of $X$ such that:

(1) if $A, B \in F$ then $A \cap B \in F$,

(2) if $A \subset B$ and $A \in F$ then $B \in F$.

For example, given a point, the collection of all neighborhoods of that point is a filter.

A filter is said to be convergent to a point if any neighborhood of that point is an element of the filter.

A filter-base (co sở lọc) is a collection $G$ of non-empty subsets of $X$ such that if $A, B \in G$ then there is $C \in G$ such that $G \subset (A \cap B)$. 
If $G$ is a filter-base in $X$ then the filter generated by $G$ is defined to be the collection of all subsets of $X$ each containing an element of $G$: $\{ A \subset X \mid \exists B \in G, B \subset A \}$.

For example, in a metric space, the collection of all open balls centered at a point is the filter-base for the filter consisting of all neighborhoods of that point.

A filter-base is said to be convergent to a point if the filter generated by it converges to that point.

(1) Show that a filter-base is convergent to $x$ if and only if every neighborhood of $x$ contains an element of the filter-base.

(2) Show that a point $x \in X$ is a limit point of a subspace $A$ of $X$ if and only if there is a filter-base in $A \setminus \{x\}$ convergent to $x$.

(3) Show that a map $f : X \to Y$ is continuous at $x$ if and only if for any filter-base $F$ that is convergent to $x$, the filter-base $f(F)$ is convergent to $f(x)$.

1.8. Compact Space

A cover of a set \( X \) is a collection of subsets of \( X \) whose union is \( X \). A cover is said to be an open cover if each member of the cover is an open subset of \( X \). A subset of a cover which is itself still a cover is called a subcover.

A space is \( \textit{compact} \) if every open cover has a finite subcover.

**Example 1.8.1.** Any finite space is compact.

**Remark 1.8.2.** Let \( A \) be a subspace of a topological space \( X \). Let \( I \) be an open cover of \( A \). Each \( O \in I \) is an open set of \( A \), so it is the restriction of an open set \( U_O \) of \( X \) whose union contains \( A \). On the other hand if we have a collection \( I \) of open sets of \( X \) whose union contains \( A \) then the collection \( \{ U \cap O \mid O \in I \} \) is an open cover of \( A \). For this reason we often use the term open cover of a subspace \( A \) of \( X \) in both senses: either as an open cover of \( A \) or as a collection of open cover of the space \( X \) whose union contains \( A \).

**Example 1.8.3.** Any subset of \( \mathbb{R} \) with the finite complement topology is compact. Note that this space is not Hausdorff (1.6.6).

**Theorem 1.8.4.** In a Hausdorff space compact subsets are closed.

**Proof.** Let \( A \) be a compact set in a Hausdorff space \( X \). We show that \( X \setminus A \) is open.

Let \( x \in X \setminus A \). For each \( a \in A \) there are disjoint open sets \( U_a \) containing \( x \) and \( V_a \) containing \( a \). The collection \( \{ V_a \mid a \in A \} \) covers \( A \), so there is a finite subcover \( \{ V_{a_i} \mid 1 \leq i \leq n \} \). Let \( U = \bigcap_{i=1}^n U_{a_i} \) and \( V = \bigcup_{i=1}^n V_{a_i} \). Then \( U \) is an open neighborhood of \( x \) disjoint from \( V \), a neighborhood of \( A \).

**Theorem 1.8.5 (Continuous image of compact space is compact).** If \( X \) is compact and \( f : X \to Y \) is continuous then \( f(X) \) is compact.

**Proposition 1.8.6.** A closed subspace of a compact space is compact.

**Proof.** Suppose that \( X \) is compact and \( A \subset X \) is closed. Let \( I \) be an open cover of \( A \). By adding the open set \( X \setminus A \) to \( I \) we get an open cover of \( X \). This open cover has a finite subcover. This subcover of \( X \) must contain \( X \setminus A \), thus omitting this set we get a finite subcover of \( A \) from \( I \).

**Proposition 1.8.7.** In a compact space an infinite set has a limit point.

**Proof.** Let \( A \) be an infinite set in a compact space \( X \). Suppose that \( A \) has no limit point. Let \( x \in X \), then there is an open neighborhood \( U_x \) of \( x \) that contains at most one point of \( A \). The collection of such \( U_x \) cover \( A \), so there is a finite subcover. But that implies that \( A \) is finite.
Characterization of compact spaces in terms of closed subsets. In the definition of compact spaces, writing open sets as complements of closed sets, we get a dual statement: A space is compact if for every collection of closed subsets whose intersection is empty there is a a finite subcollection whose intersection is empty.

A collection of subsets of a set is said to have the finite intersection property if the intersection of every finite subcollection is non-empty.

**Theorem 1.8.8.** A space is compact if and only if every collection of closed subsets with the finite intersection property has non-empty intersection.

**Compact sets in metric spaces.** The following results have been studied in Functional Analysis. It is useful for the reader to review them.

**Proposition 1.8.9.** A metric space is compact if and only if every sequence has a convergent subsequence.

**Proposition 1.8.10.** If a subset of a metric space is compact then it is closed and bounded.

**Proposition 1.8.11.** A subset of the Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

**Example 1.8.12.** In the Euclidean space $\mathbb{R}^n$ the closed ball $B'(a, r)$ is compact.

**Theorem 1.8.13 (Lebesgue’s number).** Let $X$ be a compact metric space and let $I$ be an open cover of $X$. Then there exists a number $\epsilon > 0$ such that any ball $B(x, \epsilon)$ is contained in a member of the cover $I$.

**Proof.** Given $x \in X$ there is an open set $U_x \in I$ containing $x$. There is a number $\epsilon_x > 0$ such that the ball $B(x, 2\epsilon_x)$ is contained in $U_x$. The collection \{\(B(x, \epsilon_x) \mid x \in X\}\} is an open cover of $X$, therefore there is a finite subcover \{\(B(x_i, \epsilon_i) \mid 1 \leq i \leq n\}\}. Let $\epsilon = \min\{\epsilon_i \mid 1 \leq i \leq n\}$.

Suppose that $y \in B(x, \epsilon)$. There is an $i_0, 1 \leq i_0 \leq n$, such that $x \in B(x_{i_0}, \epsilon_{i_0})$. We have $d(y, x_{i_0}) < \epsilon + \epsilon_{i_0} \leq 2\epsilon$. This implies $y$ belongs to the element $U_{x_{i_0}}$ of $I$, and so $B(x, \epsilon)$ is contained in $U_{x_{i_0}}$.

**Problems.**

1.8.14. Find a cover of the interval $(0, 1)$ with no finite subcover.

1.8.15. A discrete compact topological space is finite.

1.8.16. In a topological space a finite unions of compact subsets is compact.

1.8.17. In a Hausdorff space an intersection of compact subsets is compact.

1.8.18 (An extension of Cantor Lemma in Calculus). Let $X$ be compact and $X \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \ldots$ be a ascending sequence of closed, non-empty sets. Then $\bigcap_{i=1}^{\infty} A_n \neq \emptyset$.

1.8.19 (The extreme value theorem). * If $X$ is a compact space and $f : X \to (\mathbb{R}, \text{Euclidean})$ is continuous then $f$ has a maximum value and a minimum value.
1.8.20. * If $X$ is compact, $Y$ is Hausdorff, $f : X \to Y$ is bijective and continuous, then $f$ is a homeomorphism.

1.8.21. In a Hausdorff space a point and a disjoint compact set can be separated by open sets.

*Hint:* See the proof of 1.8.4.

1.8.22. In a regular space a closed set and a disjoint compact set can be separated by open sets.

1.8.23. In a Hausdorff space two disjoint compact sets can be separated by open sets.

*Hint:* Use 1.8.21.

1.8.24. A compact Hausdorff space is normal.

*Hint:* Use 1.8.23.

1.8.25. * The set of $n \times n$-matrix with real coefficients, denoted by $M(n; \mathbb{R})$, could be naturally considered as a subset of the Euclidean space $\mathbb{R}^{n^2}$ by considering entries of a matrix as coordinates, via the map

$$ (a_{i,j}) \mapsto (a_{1,1}, a_{2,1}, \ldots, a_{n,1}, a_{1,2}, a_{2,2}, \ldots, a_{n,2}, a_{1,3}, \ldots, a_{n-1,n}, a_{n,n}). $$

The **Orthogonal Group** $O(n)$ is defined to be the group of matrices representing orthogonal linear maps of $\mathbb{R}^n$, that are linear maps that preserve inner product. Thus

$$ O(n) = \{A \in M(n; \mathbb{R}) | A \cdot A^T = I_n\}. $$

The **Special Orthogonal Group** $SO(n)$ is the subgroup of $O(n)$ consisting of all orthogonal matrices with determinant 1.

1. (1) Show that any element of $SO(2)$ is of the form

$$ \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\
\sin(\varphi) & \cos(\varphi) \end{pmatrix}. $$

This is a rotation in the plane around the origin with an angle $\varphi$. Thus $SO(2)$ is the group of rotations on the plane around the origin.

(2) Show that $SO(2)$ is path-connected.

(3) How many connected components does $O(2)$ have?

(4) Show that $SO(n)$ is compact.

(5) The **General Linear Group** $GL(n; \mathbb{R})$ is the group of all invertible $n \times n$-matrices with real coefficients. Show that $GL(n; \mathbb{R})$ is not compact.

(6) Find the number of connected components of $GL(n; \mathbb{R})$.

1.8.26 (Point-wise convergence topology). Let $X$ and $Y$ be two spaces. Let $(f_i)_{i \in I}$ be a net of functions from $X$ to $Y$.

We say that the net $(f_i)_{i \in I}$ converges to a function $f : X \to Y$ point-wise if for any $x \in X$ the net $(f_i(x))_{i \in I}$ converges to $f(x)$.

Now we consider a function from $X$ to $Y$ as an element of the set $Y^X = \prod_{x \in X} Y$. In this view a function $f : X \to Y$ is an element $f \in Y^X$, and for each $x \in X$ the value $f(x)$ is the coordinate of the element $f$.

1. (1) Show that with this point of view, the net of functions $(f_i)_{i \in I}$ converges to a function $f : X \to Y$ point-wise if and only if the net of points $(f_i)_{i \in I}$ converges to the
point $f$ in the product topology of $Y^X$. (b) Let $C(X,Y)$ be the set of all continuous functions from $X$ to $Y$. Define the point-wise convergence topology on $C(X,Y)$ as the topology generated by sets of the form

$$S(x,U) = \{ f \in C(X,Y) \mid f(x) \in U \}$$

with $x \in X$ and $U \subset Y$ is open.

(2) Show that the point-wise convergence topology is exactly the product topology on $Y^X$ restricted to $C(X,Y)$.

*Hint:* See 1.9.6 and 1.9.9.

1.8.27 (Compact-open topology). * Let $X$ and $Y$ be two spaces. Let $C(X,Y)$ be the set of all continuous functions from $X$ to $Y$.

The topology on $C(X,Y)$ generated by all sets of the form

$$S(A,U) = \{ f \in C(X,Y) \mid f(A) \subset U \}$$

where $A \subset X$ is compact and $U \subset Y$ is open is called the compact-open topology on $C(X,Y)$.

(1) A net $(f_i)$ in $C(X,Y)$ is said to be convergent point-wise (hội tụ từng điểm) to $f \in C(X,Y)$ if for all $x \in X$ the net $(f_i(x))$ converges to $f(x)$.

(2) Show that if $(f_i)$ converges to $f$ in the compact-open topology then it converges to $f$ point-wise.

(3) Show that if $(f_i)$ converges to $f$ in the compact-open topology then $f$ is continuous.

(4) If $X$ is compact and $Y$ is a metric space with a metric $d$ then we say the net $(f_i)$ converges to $f$ uniformly if the net $(\sup\{d(f_i(x), f(x)) \mid x \in X\})$ converges to 0.

This is the usual uniform convergence in Analysis.

(5) Show that if $(f_i)$ converges to $f$ in the compact-open topology then it converges to $f$ uniformly.

(6) Define a metric on $C(X,Y)$ as $d(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}$. This metric generates a topology on $C(X,Y)$ called the uniform convergence topology. In fact, the compact-open topology is exactly the uniform convergence topology.
1. GENERAL TOPOLOGY

1.9. Product of Spaces

**Finite products of spaces.** Let $X$ and $Y$ be two topological spaces, and consider the Cartesian product $X \times Y$. The *product topology* on $X \times Y$ is the topology generated by the collection $F$ of sets of the form $U \times V$ where $U$ is an open set of $X$ and $V$ is an open set of $Y$.

The collection $F$ is a subbasis for the product topology. Actually, since the intersection of two members of $F$ is also a member of $F$, the collection $F$ is a basis for the product topology.

Thus every open set in the product topology is a union of products of open sets of $X$ with open sets of $Y$.

Similarly, the product topology on $\prod_{i=1}^{n} (X_i, \tau_i)$ is defined to be the topology generated by the collection $\prod_{i=1}^{n} \tau_i$.

**Remark 1.9.1.** Note that, as sets:

1. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
2. $(A \times B) \cup (C \times D) \subsetneq (A \cup C) \times (B \cup D) = (A \times B) \cup (A \times D) \cup (C \times B) \cup (C \times D)$.

**Proposition 1.9.2.** If each $b_i$ is a basis for $X_i$ then $\prod_{i=1}^{n} b_i$ is a basis for the product topology on $\prod_{i=1}^{n} X_i$.

**Example 1.9.3 (Euclidean topology).** Recall that $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, $n$ times. The Euclidean topology on $\mathbb{R}^n$ is generated by open intervals. An open set in the product topology of $\mathbb{R}^n$ is a union of products of open intervals.

Since a product of open intervals is an open rectangle, and an open rectangle is a union of open balls and vice versa, the product topology is exactly the Euclidean topology (see 1.2.12).

**Infinite products of spaces.**

**Definition 1.9.4.** Let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of topological spaces. The *product topology* on $\prod_{i \in I} X_i$ is the topology generated by the collection $F$ consisting of all sets of the form $\prod_{i \in I} U_i$, where $U_i \in \tau_i$ and $U_i = X_i$ for all except finitely many $i \in I$.

**Proposition 1.9.5.** The collection $F$ above is a basis of the product topology.

For $j \in I$ the projection $p_j : \prod_{i \in I} X_i \to X_j$ is defined by $p_j((x_i)) = x_j$. It is the “projection to the $j$ coordinate”.

The definition of the product topology is explained in the following:

**Theorem 1.9.6 (Product topology is the topology such that projections are continuous).** The product topology is the coarsest topology on $\prod_{i \in I} X_i$ such that all the projection maps $p_i$ are continuous. In other words, the product topology is the topology generated by the projection maps.
1.9. PRODUCT OF SPACES

Note that if \( O_j \in X_j \) then \( p^{-1}_j(O_j) = \prod_{i \in I} U_i \) with \( U_i = X_i \) for all \( i \) except \( j \), and \( U_j = O_j \).

The topology generated by all the maps \( p_i \) is the topology generated by all sets of the form \( p^{-1}_i(O_i) \) with \( O_i \in \tau_i \), see 1.3.13. A finite intersection of these sets is exactly a member of the basis of the product topology as in the definition. \( \square \)

**Theorem 1.9.7 (Map to product space is continuous if and only if each component map is continuous).** A map \( f: Y \to \prod_{i \in I} X_i \) is continuous if and only if each component \( f_i = p_i \circ f \) is continuous.

**Remark 1.9.8.** However continuity of a map from a product space is not the same as continuity with respect to each variable, as we have seen in Calculus.

**Theorem 1.9.9 (Convergence in product topology is coordinate-wise convergence).** A net \( n: J \to \prod_{i \in I} X_i \) is convergent if and only if all of its projections \( p_i \circ n \) are convergent.

**Proof.** \( (\Leftarrow) \) Suppose that each \( p_i \circ n \) is convergent to \( a_i \), we show that \( n \) is convergent to \( a = (a_i)_{i \in I} \).

A neighborhood of \( a \) contains an open set of the form \( U = \prod_{i \in I} O_i \) with \( O_i \) are open sets of \( X_i \) and \( O_i = X_i \) except for \( i \in K \), where \( K \) is a finite subset of \( I \).

For each \( i \in K \), \( p_i \circ n \) is convergent to \( a_i \), therefore there exists an index \( j_i \in J \) such that for \( j \geq j_i \) we have \( p_i(n(j)) \in O_i \). Take an index \( j_0 \) such that \( j_0 \geq j_i \) for all \( i \in K \). Then for \( j \geq j_0 \) we have \( n(j) \in U \). \( \square \)

**Tikhonov Theorem.**

**Theorem 1.9.10 (Tikhonov theorem).** A product of compact spaces is compact.

**Example 1.9.11.** Let \( [0, 1] \) have the Euclidean topology. The space \( \prod_{i \in \mathbb{Z}^+} [0, 1] \) is called the Hilbert cube. By Tikhonov theorem the Hilbert cube is compact.

Applications of Tikhonov theorem include the Banach-Alaoglu theorem in Functional Analysis and the Stone-Cech compactification.

Tikhonov theorem is equivalent to the Axiom of choice. The proofs we have are rather difficult. However in the case of finite product it can be proved more easily (1.9.25). Different techniques can be used in special cases of this theorem (1.9.30 and 1.9.31).

**Proof of Tikhonov theorem.** Let \( X_i \) be compact for all \( i \in I \). We will show that \( X = \prod_{i \in I} X_i \) is compact by showing that if a collection of closed subsets of \( X \) has the finite intersection property then it has non-empty intersection (see 1.8.8).

Let \( F \) be a collection of closed subsets of \( X \) that has the finite intersection property. We will show that \( \bigcap_{A \in F} A \neq \emptyset \).

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10 Proved by Andrei Nicolae Tihonov around 1926. The product topology was defined by him. His name is also spelled as Tichonoff.

11 A proof based on open covers is also possible, see [Kel55] p. 143.
Have a look at the following argument, which suggests that proving the Tikhonov theorem might not be easy. If we take the closures of the projections of the collection \( F \) to the \( i \)-coordinate then we get a collection \( \{ p_i(A), A \in F \} \) of closed subsets of \( X_i \) having the finite intersection property. Since \( X_i \) is compact, this collection has non-empty intersection.

From this it is tempting to conclude that \( F \) must have non-empty intersection itself. But that is not true, see the figure.

In what follows we will overcome this difficulty by first enlarging the collection \( F \).

(1) We show that there is a maximal collection \( \hat{F} \) of subsets of \( X \) such that \( \hat{F} \) contains \( F \) and still has the finite intersection property. We will use Zorn lemma for this purpose.\(^{12}\)

Let \( K \) be the collection of collections \( G \) of subsets of \( X \) such that \( G \) contains \( F \) and has the finite intersection property. On \( K \) we define an order by the usual set inclusion.

Now suppose that \( L \) is a totally ordered subcollection of \( K \). Let \( H = \bigcup_{G \in L} G \). We will show that \( H \in K \), therefore \( H \) is an upper bound of \( L \).

First \( H \) contains \( F \). We need to show that \( H \) has the finite intersection property. Suppose that \( H_i \in H, 1 \leq i \leq n \). Then \( H_i \in G_i \) for some \( G_i \in L \).

Since \( L \) is totally ordered, there is an \( i_0, 1 \leq i_0 \leq n \) such that \( G_{i_0} \) contains all \( G_i, 1 \leq i \leq n \). Then \( H_i \in G_{i_0} \) for all \( 1 \leq i \leq n \), and since \( G_{i_0} \) has the finite intersection property, we have \( \bigcap_{i=1}^n H_i \neq \emptyset \).

(2) Since \( \hat{F} \) is maximal, it is closed under finite intersection. Moreover if a subset of \( X \) has non-empty intersection with every element of \( \hat{F} \) then it belongs to \( \hat{F} \).

(3) Since \( \hat{F} \) has the finite intersection property, for each \( i \in I \) the collection \( \{ p_i(A) \mid A \in \hat{F} \} \) also has the finite intersection property, and so does the collection \( \{ \overline{p_i(A)} \mid A \in \hat{F} \} \). Since \( X_i \) is compact, \( \bigcap_{A \in \hat{F}} p_i(A) \) is non-empty.

(4) Let \( x_i \in \bigcap_{A \in \hat{F}} p_i(A) \) and let \( x = (x_i)_{i \in I} \in \prod_{i \in I} [\bigcap_{A \in \hat{F}} p_i(A)] \). We will show that \( x \in \overline{A} \) for all \( A \in \hat{F} \), in particular \( x \in A \) for all \( A \in F \).

We need to show that any neighborhood of \( x \) has non-empty intersection with every \( A \in \hat{F} \). It is sufficient to prove this for neighborhoods of \( x \) belonging to the basis of \( X \), namely finite intersections of sets of the form

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\(^{12}\)This is a routine step; it might be easier for the reader to carry it out instead of reading.
$p^{-1}_i(O_i)$ where $O_i$ is an open neighborhood of $x_i = p_i(x)$. For any $A \in \tilde{F}$, since $x_i \in p_i(A)$ we have $O_i \cap p_i(A) \neq \emptyset$. Therefore $p^{-1}_i(O_i) \cap A \neq \emptyset$. By the maximality of $\tilde{F}$ we have $p^{-1}_i(O_i) \in \tilde{F}$, and the desired result follows.

Problems.

1.9.12. Check that in topological sense $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

* Fix a point $O = (O_i) \in \prod_{i \in I} X_i$. Define the inclusion map $f : X_i \to \prod_{i \in I} X_i$ by

$$x \mapsto f(x) \text{ with } f(x)_j = \begin{cases} O_j & \text{if } j \neq i \\ x & \text{if } j = i \end{cases}.$$  

Show that $f$ is a homeomorphism onto its image $\tilde{X}_i$ (an embedding of $X_i$).

Thus $\tilde{X}_i$ is a copy of $X_i$ in $\prod_{i \in I} X_i$. The spaces $\tilde{X}_i$ have $O$ as the common point.

This is an analog of the coordinate system $Oxy$ on $\mathbb{R}^2$.

* Show that each projection map $p_i$ is an open map, mapping an open set onto an open set.

* Show that a space $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X\}$ is closed in $X \times X$, by:

(a) using nets,

(b) not using nets.

1.9.13. Show that the sphere $S^2$ with the North Pole and the South Pole removed is homeomorphic to the infinite cylinder $S^1 \times \mathbb{R}$.

1.9.14. Show that a space $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X\}$ is closed in $X \times X$, by:

(a) using nets,

(b) not using nets.

1.9.15. If $Y$ is Hausdorff and $f : X \to Y$ is continuous then the graph of $f$ (the set $\{(x, f(x)) | x \in X\}$) is closed in $X \times Y$.

1.9.16. If for each $i \in I$ the space $X_i$ is homeomorphic to the space $Y_i$ then $\prod_{i \in I} X_i$ is homeomorphic to $\prod_{i \in I} Y_i$.

1.9.17. Show that each projection map $p_i$ is an open map, mapping an open set onto an open set.

* Fix a point $O = (O_i) \in \prod_{i \in I} X_i$. Define the inclusion map $f : X_i \to \prod_{i \in I} X_i$ by

$$x \mapsto f(x) \text{ with } f(x)_j = \begin{cases} O_j & \text{if } j \neq i \\ x & \text{if } j = i \end{cases}.$$  

Show that $f$ is a homeomorphism onto its image $\tilde{X}_i$ (an embedding of $X_i$).

Thus $\tilde{X}_i$ is a copy of $X_i$ in $\prod_{i \in I} X_i$. The spaces $\tilde{X}_i$ have $O$ as the common point.

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1.9.18. Is the projection $p_i$ a closed map, in general?

1.9.19. Is it true that a map on a product space is continuous if it is continuous on each variable?

1.9.20 (Disjoint union). Let $A$ and $B$ be topological spaces.

On the set $(A \times \{0\}) \cup (B \times \{1\})$ consider the topology generated by subsets of the form $U \times \{0\}$ and $V \times \{1\}$ where $U$ is open in $A$ and $V$ is open in $B$. Show that $A \times \{0\}$ is homeomorphic to $A$, while $B \times \{1\}$ is homeomorphic to $B$. In this space $A \times \{0\}$ and $B \times \{1\}$ are (disjoint) connected components.

The space $(A \times \{0\}) \cup (B \times \{1\})$ is called the disjoint union (hội rời) of $A$ and $B$, denoted by $A \sqcup B$. We use this construction when for example we want to consider a space consisting of two disjoint circles.

1.9.21. Show that $f$ is a homeomorphism onto its image $\tilde{X}_i$ (an embedding of $X_i$).

Thus $\tilde{X}_i$ is a copy of $X_i$ in $\prod_{i \in I} X_i$. The spaces $\tilde{X}_i$ have $O$ as the common point.

This is an analog of the coordinate system $Oxy$ on $\mathbb{R}^2$.
1.9.22. *(a) If each \( X_i \), \( i \in I \) is Hausdorff then \( \prod_{i \in I} X_i \) is Hausdorff.
(b) If \( \prod_{i \in I} X_i \) is Hausdorff then each \( X_i \) is Hausdorff.
\[\text{Hint: use 1.9.21}\]

1.9.23. *(a) If \( \prod_{i \in I} X_i \) is path-connected then each \( X_i \) is path-connected.
(b) If each \( X_i, i \in I \) is path-connected then \( \prod_{i \in I} X_i \) is path-connected.
\[\text{Hint: use 1.9.21}\]

1.9.24. *(a) If \( \prod_{i \in I} X_i \) is connected then each \( X_i \) is connected.
(b) If \( X \) and \( Y \) are connected then \( X \times Y \) is connected.
\[\text{Hint: use 1.9.21}\]

1.9.25. *(a) If \( \prod_{i \in I} X_i \) is compact then each \( X_i \) is compact.
(b) If \( X \) and \( Y \) are compact then \( X \times Y \) is compact.
\[\text{Hint: use 1.9.21}\]

1.9.26. *(a) If \( O_i \) is an open set in \( X_i \) for all \( i \in I \) then is \( \prod_{i \in I} O_i \) open?
(b) If \( F_i \) is a closed set in \( X_i \) for all \( i \in I \) then is \( \prod_{i \in I} F_i \) closed?

1.9.27. Consider the Euclidean space \( \mathbb{R}^n \). Show that the usual addition \((x, y) \mapsto x + y\) is a continuous map from \( \mathbb{R}^n \times \mathbb{R}^n \) to \( \mathbb{R}^n \), while the usual scalar multiplication \((c, x) \mapsto c \cdot x\) is a continuous map from \( \mathbb{R} \times \mathbb{R}^n \) to \( \mathbb{R}^n \).

This is an example of a topological vector space.

1.9.28. Consider the Euclidean space \( \mathbb{R}^n \). Show that the usual addition \((x, y) \mapsto x + y\) and the usual subtraction \((x, y) \mapsto x - y\) are continuous maps from \( \mathbb{R}^n \times \mathbb{R}^n \) to \( \mathbb{R}^n \).

Notice that with those two operations \( \mathbb{R}^n \) is a group. This is an example of a topological group.

1.9.29 (Zariski topology). Let \( F = \mathbb{R} \) or \( F = \mathbb{C} \).

A polynomial in \( n \) variables on \( F \) is a function from \( F^n \) to \( F \) that is a finite sum of terms of the form \( a x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \), where \( a, x_i \in F \) and \( m_i \in \mathbb{N} \). Let \( P \) be the set of all polynomials in \( n \) variables on \( F \).

If \( S \subset P \) then define \( Z(S) \) to be the set of all common zeros of all polynomials in \( S \), thus \( Z(S) = \{ x \in F^n \mid \forall p \in S, p(x) = 0 \} \). Such a set is called an algebraic set.

(a) Show that if we define that a subset of \( F^n \) is closed if it is algebraic, then this gives a topology on \( F^n \), called the Zariski topology.
(b) Show that the Zariski topology on \( F \) is exactly the finite complement topology.
(c) Show that if both \( F \) and \( F^n \) have the Zariski topology then all polynomials on \( F^n \) are continuous.
(d) Is the Zariski topology on \( F^n \) the product topology?

The Zariski topology is used in Algebraic Geometry.
1.9.30. Using the characterization of compact subsets of Euclidean spaces, prove the Tikhonov theorem for finite products of compact subsets of Euclidean spaces.

1.9.31. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Prove that the product topology on \(X \times Y\) is given by the metric
\[
d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.
\]

Using the characterization of compact metric spaces in terms of sequences, prove the Tikhonov theorem for finite products of compact metric spaces.

Further readings

Strategy for a proof of Tikhonov theorem based on net. The proof that we will outline here is based on further developments of the theory of nets and a characterization of compactness in terms of nets.

Definition 1.9.32 (Subnet). Let \(I\) and \(I'\) be directed sets, and let \(h : I' \rightarrow I\) be a map such that
\[
\forall k \in I, \exists k' \in I', (i' \geq k' \Rightarrow h(i') \geq k).
\]
If \(n : I \rightarrow X\) is a net then \(n \circ h\) is called a subnet of \(n\).

The notion of subnet is an extension of the notion of subsequence. If we take \(n_i \in \mathbb{Z}^+\) such that \(n_i < n_{i+1}\) then \((x_n)\) is a subsequence of \((x_n)\). In this case the map \(h : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+\) given by \(h(i) = n_i\) is a strictly increasing function. Thus a subsequence of a sequence is a subnet of that sequence. On the other hand a subnet of a sequence does not need to be a subsequence, since for a subnet the map \(h\) is only required to satisfy \(\lim_{i \to \infty} h(i) = \infty\).

A net \((x_i)_{i \in I}\) is called eventually in \(A \subset X\) if there is \(j \in I\) such that \(i \geq j \Rightarrow x_i \in A\).

Definition 1.9.33. Universal net A net \(n\) in \(X\) is universal if for any subset \(A\) of \(X\) either \(n\) is eventually in \(A\) or \(n\) is eventually in \(X\setminus A\).

Proposition 1.9.34. If \(f : X \rightarrow Y\) is continuous and \(n\) is a universal net in \(X\) then \(f(n)\) is a universal net.

Theorem 1.9.35. The following statements are equivalent:

1. \(X\) is compact.
2. Every universal net in \(X\) is convergent.
3. Every net in \(X\) has a convergent subnet.

The proof of the last two propositions above could be found in [Bre93]. Then we finish the proof of Tikhonov theorem as follows.

Proof of Tikhonov Theorem. Let \(X = \prod_{i \in I} X_i\) where each \(X_i\) is compact. Suppose that \((x_j)_{j \in J}\) is a universal net in \(X\). By 1.9.9 the net \((x_j)\) is convergent if and only if the projection \((p_i(x_j))\) is convergent for all \(i\). But that is true since \((p_i(x_j))\) is a universal net in the compact set \(X_i\). \(\square\)
1.10. Compactification

A compactification (compắc hóa) of a space $X$ is a compact space $Y$ such that $X$ is homeomorphic to a dense subset of $Y$.

**Example 1.10.1.** A compactification of the Euclidean interval $(0, 1)$ is the Euclidean interval $[0, 1]$. Another is the circle $S^1$. Yet another is the Topologist’s sine curve $\{(x, \sin \frac{1}{x}) \mid 0 < x \leq 1 \} \cup \{(0, y) \mid -1 \leq y \leq 1 \}$ (see 1.5.1).

**Example 1.10.2.** A compactification of the complex plane $\mathbb{C}$ is the sphere $S^2$, often called the Riemann sphere.

**One-point compactification.** In some cases it is possible to compactify a non-compact space by adding just one point. It is called a one-point compactification. For example $[0, 1]$ is a compactification of $(0, 1)$.

Let $X$ be a space, let $\infty$ be not in $X$, and let $X^\infty = X \cup \{\infty\}$. Let us see what a topology on $X^\infty$ should be in order for $X^\infty$ to be compact. First $X^\infty$ contains $X$ as a subspace, so each open subset of $X$ must be open in $X^\infty$. If an open subset of $X^\infty$ contains $\infty$ then its complement in $X^\infty$ must be closed in $X^\infty$, and hence is compact, and is a subset of $X$.

The existence of such a topology is given in the following:

**Theorem 1.10.3 (Alexandroff compactification).** Let $X$ be a space. Let $\infty$ be not in $X$ and let $X^\infty = X \cup \{\infty\}$. Define a topology on $X^\infty$ as follow: an open set in $X^\infty$ is an open subset of $X$, or is $X^\infty \setminus C$ where $C$ is a closed compact subset of $X$. With this topology $X^\infty$ is compact and contains $X$ as a subspace.

If $X$ is not compact then $X$ is dense in $X^\infty$, and in this case $X^\infty$ is called the Alexandroff compactification of $X$.

**Proof.** We go through several steps.

1. We check that we really have a topology.

   Let $\{C_i\}_{i \in I}$ be a collection of closed compact sets in $X$. Then $\bigcup_i (X^\infty \setminus C_i) = X^\infty \setminus \bigcap_i C_i$, where $\bigcap_i C_i$ is closed compact.

   If $O$ is open in $X$ and $C$ is closed compact in $X$ then $O \cup (X^\infty \setminus C) = X^\infty \setminus (C \setminus O)$, where $C \setminus O$ is closed and compact subset of $X$.

   Also $O \cap (X^\infty \setminus C) = O \cap (X \setminus C)$ is open in $X$.

   If $C_1$ and $C_2$ are closed compact in $X$ then $(X^\infty \setminus C_1) \cap (X^\infty \setminus C_2) = X^\infty \setminus (C_1 \cup C_2)$, where $C_1 \cup C_2$ is closed compact.

   So we do have a topology. With this topology $X$ is a subspace of $X^\infty$.

2. We show that $X^\infty$ is compact. Let $F$ be an open cover of $X^\infty$. Then an element $O \in F$ will cover $\infty$. The complement of $O$ in $X^\infty$ is a closed compact set $C$ in $X$.

   Then $F \setminus \{O\}$ is an open cover of $C$. From this cover there is a finite cover. This finite cover together with $O$ is a finite cover of $X^\infty$.

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13Proved in the early 1920s by Pavel Sergeyevich Alexandrov. Alexandroff is another way to spell his name.
(3) Since $X$ is not compact and $X^\infty$ is compact, $X$ cannot be closed in $X^\infty$, therefore the closure of $X$ in $X^\infty$ is $X^\infty$. □

A space $X$ is called \textit{locally compact} if every point has a compact neighborhood.

**Example 1.10.4.** The Euclidean space $\mathbb{R}^n$ is locally compact.

**Proposition 1.10.5.** The Alexandroff compactification of a locally compact Hausdorff space is Hausdorff.

\textbf{Proof.} Suppose that $X$ is locally compact and is Hausdorff. We check that $\infty$ and $x \in X$ can be separated by open sets. Since $X$ is locally compact there is a compact set $C$ containing an open neighborhood $O$ of $x$. Since $X$ is Hausdorff, $C$ is closed in $X$. Then $X^\infty \setminus C$ is open in the Alexandroff compactification $X^\infty$. So $O$ and $X^\infty \setminus C$ separates $x$ and $\infty$. □

The need for the locally compact assumption is discussed in 1.10.20.

**Proposition 1.10.6.** If $X$ is homeomorphic to $Y$ then a Hausdorff one-point compactification of $X$ is homeomorphic to a Hausdorff one-point compactification of $Y$.

In particular, Hausdorff one-point compactification is unique up to homeomorphisms. For this reason sometimes we mention \textit{the} one-point compactification.

\textbf{Proof.} Suppose that $h : X \to Y$ is a homeomorphism. Let $X \cup \{a\}$ and $Y \cup \{b\}$ be Hausdorff one-point compactifications of $X$ and $Y$. Let $\tilde{h} : X \cup \{a\} \to Y \cup \{b\}$ be defined by $\tilde{h}(x) = h(x)$ if $x \neq a$ and $\tilde{h}(a) = b$. We show that $\tilde{h}$ is a homeomorphism. Because the role of $X$ and $Y$ are same, it is sufficient to prove that $\tilde{h}$ is continuous, or we can refer to 1.8.20.

Let $U$ be an open subset of $Y \cup \{b\}$. If $U$ does not contain $b$ then $U$ is open in $Y$, so $h^{-1}(U)$ is open in $X$, and so is open in $X \cup \{a\}$. If $U$ contains $b$ then $(Y \cup \{b\}) \setminus U$ is closed in $Y \cup \{b\}$, which is compact, so $(Y \cup \{b\}) \setminus U = Y \setminus U$ is compact. Then $\tilde{h}^{-1}((Y \cup \{b\}) \setminus U) = h^{-1}(Y \setminus U)$ is a compact subspace of $X \cup \{a\}$. Since $X \cup \{a\}$ is Hausdorff, $\tilde{h}^{-1}((Y \cup \{b\}) \setminus U)$ is closed in $X \cup \{a\}$. Thus $\tilde{h}^{-1}(U)$ must be open in $X \cup \{a\}$. □

**Example 1.10.7.** The Euclidean line $\mathbb{R}$ is homeomorphic to the circle $S^1$ minus a point. The circle is of course a Hausdorff one-point compactification of the circle minus a point. Thus a Hausdorff one-point compactification (in particular, the Alexandroff compactification) of the Euclidean line is homeomorphic to the circle.

**Stone-Cech compactification.** Let $X$ be a space. Denote by $C(X)$ the set of all bounded continuous functions from $X$ to $\mathbb{R}$ where $\mathbb{R}$ has the Euclidean topology.

Let $\Phi$ be a function from $X$ to $Y = \prod_{f \in C(X)} [\inf f, \sup f]$ defined by for each $x \in X$ and each $f \in C(X)$, the $f$-coordinate of the point $\Phi(x)$ is $\Phi(x)_f = f(x)$. This means the $f$-component of $\Phi$ is $f$, i.e. $p_f \circ \Phi = f$, where $p_f$ is the projection to the $f$-coordinate.
Since $Y$ is compact, the subspace $\Phi(X)$ is compact. Under certain conditions $\Phi$ is an embedding, and so the space $\Phi(X)$ is a compactification of $X$.

**Definition 1.10.8.** A space is said to be completely regular (also called a $T_{3\frac{1}{2}}$-space) if it is a $T_1$-space and for each point $x$ and each closed set $A$ with $x \notin A$ there is a map $f \in C(X)$ such that $f(x) = a$ and $f(A) = \{b\}$ where $a \neq b$.

Thus in a completely regular space a point and a closed set disjoint from it can be separated by a continuous real function.

**Theorem 1.10.9.** If $X$ is completely regular then $\Phi : X \to \Phi(X)$ is a homeomorphism. In other words $\Phi$ is an embedding.

In this case the space $\Phi(X)$ is called the Stone-Čech compactification of $X$.

**Proof.** We goes through several steps.

1. $\Phi$ is injective: If $x \neq y$ then since $X$ is completely regular there is $f \in C(X)$ such that $f(x) \neq f(y)$, therefore $\Phi(x) \neq \Phi(y)$.

2. $\Phi$ is continuous: Since the $f$-component of $\Phi$ is $f$, which is continuous, the result follows from 1.9.7.

3. $\Phi^{-1}$ is continuous: We prove that $\Phi$ brings an open set onto an open set. Let $U$ be an open subset of $X$ and let $x \in U$. There is a function $f \in C(X)$ that separates $x$ and $X \setminus U$. In particular there is an interval $(a, b)$ such that $f(x) \in ((a, b))$ and $f^{-1}((a, b)) \cap (X \setminus U) = \emptyset$. We have $f^{-1}((a, b)) = (p_f \circ \Phi)^{-1}((a, b)) = \Phi^{-1}(p_f^{-1}((a, b))) \subset U$. Apply $\Phi$ to both sides, we get $p_f^{-1}((a, b)) \cap \Phi(X) \subset \Phi(U)$. Since $p_f^{-1}((a, b)) \cap \Phi(X)$ is an open set in $\Phi(X)$ containing $\Phi(x)$, we see that $\Phi(x)$ is an interior point of $\Phi(U)$. We conclude that $\Phi(U)$ is open.

**Proposition 1.10.10.** The Stone-Čech compactification of a completely regular space is Hausdorff.

**Proof.** The space $Y$ is Hausdorff, by 1.9.22 Then the result follows from 1.6.13.

**Theorem 1.10.11.** A bounded continuous real function on a completely regular space has a unique extension to the Stone-Čech compactification of the space.

More concisely, if $X$ is a completely regular space and $f \in C(X)$ then there is a unique function $\tilde{f} \in C(\Phi(X))$ such that $f = \tilde{f} \circ \Phi$.

**Proof.** A continuous extension of $f$, if exists, is unique, by 1.7.14

Since $p_f \circ \Phi = f$ the obvious choice for $\tilde{f}$ is the projection $p_f$. 

□
Problems.

1.10.12. Find the one-point compactification of the Euclidean $(0, 1) \cup (2, 3)$.

1.10.13. What is the one-point compactification of $\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \}$ under the Euclidean topology?

1.10.14. What is the one-point compactification of $\mathbb{Z}^+$ under the Euclidean topology? How about $\mathbb{Z}$?

1.10.15. Show that $\mathbb{Q}$ is not locally compact (under the Euclidean topology of $\mathbb{R}$). Is its Alexandroff compactification Hausdorff?

1.10.16. What is the one-point compactification of the Euclidean open ball $B(0, 1)$? Find the one-point compactification of the Euclidean space $\mathbb{R}^n$.

1.10.17. What is the one-point compactification of the Euclidean annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$?

1.10.18. Define a topology on $\mathbb{R} \cup \{\pm \infty\}$ such that it is a compactification of the Euclidean $\mathbb{R}$.

1.10.19. If a subset of $X$ is closed will it be closed in the Alexandroff compactification of $X$?

1.10.20. If there is a topology on the set $X^\infty = X \cup \{\infty\}$ such that it is compact, Hausdorff, and containing $X$ as a subspace, then $X$ must be Hausdorff, locally compact, and there is only one such topology—the topology of the Alexandroff compactification.

1.10.21. We could have noticed that the notion of local compactness as we have defined is not apparently a local property. For a property to be local, every neighborhood of any point must contain a neighborhood of that point with the given property (as in the cases of local connectedness and local path-connectedness).

Show that for Hausdorff spaces local compactness is indeed a local property.

1.10.22. A locally compact Hausdorff space is regular.

Hint: Suppose that a point $x$ and a closed set $C$ are disjoint. The set $X \setminus C$ contains a compact neighborhood $A$ of $x$, by 1.10.21. So $A$ contains an open neighborhood $U$ of $x$. Since $X$ is Hausdorff, $A$ is closed.

1.10.23. A space is locally compact Hausdorff if and only if it is homeomorphic to an open subspace of a compact Hausdorff space.

Hint: $(\Rightarrow)$ Use 1.8.24 and 1.6.9.

1.10.24. A completely regular space is regular.

1.10.25. Prove 1.10.9 using nets.

1.10.26. A space is completely regular if and only if it is homeomorphic to a subspace of a compact Hausdorff space.

As a corollary, a locally compact Hausdorff space is completely regular.

Hint: $(\Rightarrow)$ Use 1.8.24 and the Urysohn lemma 1.11.1.

Further readings
By 1.10.26 if a space has a Hausdorff Alexandroff compactification then it also has a Hausdorff Stone-Cech compactification.

In a certain sense, for a noncompact space the Alexandroff compactification is the “smallest” Hausdorff compactification of the space and the Stone-Cech compactification is the “largest” one. For more discussions on this topic see for instance [Mun00, p. 237].
1.11. Urysohn Lemma

Here we consider real functions, i.e. maps to the Euclidean \( \mathbb{R} \).

Urysohn lemma.

**Theorem 1.11.1 (Urysohn lemma).** If \( X \) is normal, \( F \) is closed, \( U \) is open, and \( F \subset U \), then there exists a continuous map \( f : X \to [0, 1] \) such that \( f(x) = 0 \) on \( F \) and \( f(x) = 1 \) on \( X \setminus U \).

An equivalent statement to Urysohn lemma is:

**1.11.2.** Let \( A \) and \( B \) be two disjoint closed subsets of a normal space \( X \). Then there is a continuous function \( f \) from \( X \) to \( [0, 1] \) such that \( f(x) = 0 \) on \( A \) and \( f(x) = 1 \) on \( B \).

Thus in a normal space two disjoint closed subsets can be separated by a continuous real function.

It is much easier to prove Urysohn lemma for metric space, using the function \( f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)} \).

**Example 1.11.3.** If \( X = \mathbb{R} \), \( A = (-\infty, 0] \) and \( B = \mathbb{R} \setminus [1, \infty) \) then we can take:

\[
f(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
x & \text{if } 0 < x < 1 \\
1 & \text{if } x \geq 1 
\end{cases}
\]

**Proof of Urysohn lemma.** Recall 1.6.10 Because \( X \) is normal, if \( F \) is closed, \( U \) is open, and \( F \subset U \) then there is an open set \( V \) such that \( F \subset V \subset U \).

(1) We construct a family of open sets in the following manner.

Let \( U_1 = U \).

\[
n = 0: \; F \subset U_0 \subset \overline{U_0} \subset U_1,
\]

\[
n = 1: \; U_0 \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset U_1.
\]

\[
n = 2: \; U_0 \subset U_{\frac{1}{4}} \subset U_{\frac{1}{2}} \subset U_{\frac{3}{4}} \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{3}{4}}} \subset U_{\frac{5}{4}} \subset U_{\frac{3}{4}} \subset U_{\frac{5}{4}} \subset \cdots \subset U_{\frac{2^n-1}{2^n}} \subset \overline{U_{\frac{2^n-1}{2^n}}} \subset U_{\frac{2^n}{2^n}} = U_1.
\]

Let \( I = \{ \frac{m}{2^n} : m, n \in \mathbb{N}; 0 \leq m \leq 2^n \} \). We have a family of open sets \( \{ U_r \mid r \in I \} \) having the property \( r < s \Rightarrow U_r \subset U_s \).

(2) We can check that \( I \) is dense in \([0, 1]\) (this is really the same thing as that any real number in \([0, 1]\) can be written in binary form).

(3) Define \( f : X \to [0, 1] \),

\[
f(x) = \begin{cases} 
\inf\{ r \in I \mid x \in U_r \} & \text{if } x \in U \\
1 & \text{if } x \notin U
\end{cases}
\]
We prove that $f$ is continuous, then this is the function we are looking for. It is enough to prove that sets of the form $\{ x \mid f(x) < a \}$ and $\{ x \mid f(x) > a \}$ are open.

(4) We have $f(x) < a$ if and only if there is $r \in I$ such that $r < a$ and $x \in U_r$. Thus $\{ x \mid f(x) < a \} = \{ x \in U_r \mid r < a \} = \bigcup_{r < a} U_r$ is open.

(5) We have $f(x) > a$ if and only if there is $r \in I$ such that $r > a$ and $x \notin U_r$. Thus $\{ x \mid f(x) > a \} = \{ x \in X \setminus U_r \mid r > a \} = \bigcup_{r > a} X \setminus U_r$.

Now we show that $\bigcup_{r > a} X \setminus U_r = \bigcup_{r > a} X \setminus \bigcup_r$, which implies that $\bigcup_{r > a} X \setminus U_r$ is open. Indeed, if $r \in I$ and $r > a$ then there is $s \in I$ such that $r > s > a$. Then $\bigcup_r \subset U_r$, therefore $X \setminus U_r \subset X \setminus \bigcup_r$.

\[ \square \]

**Tietze extension theorem.** An application of Urysohn lemma is:

**Theorem 1.11.4 (Tietze extension theorem).** Let $X$ be a normal space. Let $F$ be closed in $X$. Let $f : F \to \mathbb{R}$ be continuous. Then there is a continuous map $g : X \to \mathbb{R}$ such that $g|_F = f$.

Thus in a normal space a continuous real function on a closed subspace can be extended continuously to the whole space.

**Proof.** First consider the case where $f$ is bounded.

(1) The general case can be reduced to the case when $\inf F f = 0$ and $\sup F f = 1$. We will restrict our attention to this case.

(2) By Urysohn lemma, there is a continuous function $g_1 : X \to [0, \frac{1}{2}]$ such that

\[
g_1(x) = \begin{cases} 
0 & \text{if } x \in f_0^{-1}([0, \frac{1}{3}]) \\
\frac{1}{3} & \text{if } x \in f_0^{-1}([\frac{1}{3}, 1]).
\end{cases}
\]

Let $f_1 = f - g_1$. Then $\sup_X g_1 = \frac{1}{3}$, $\sup_F f_1 = \frac{2}{3}$, and $\inf F f_1 = 0$.

(3) Inductively, once we have a function $f_n : F \to \mathbb{R}$, for a certain $n \geq 1$ we will obtain a function $g_{n+1} : X \to [0, \frac{1}{3} (\frac{2}{3})^n]$ such that

\[
g_{n+1}(x) = \begin{cases} 
0 & \text{if } x \in f_n^{-1}([0, \frac{1}{3} (\frac{2}{3})^n]) \\
\frac{1}{3} (\frac{2}{3})^n & \text{if } x \in f_n^{-1}([\frac{1}{3} (\frac{2}{3})^n, (\frac{2}{3})^n]).
\end{cases}
\]

Let $f_{n+1} = f_n - g_{n+1}$. Then $\sup_X g_{n+1} = \frac{1}{3} (\frac{2}{3})^n$, $\sup_F f_{n+1} = (\frac{2}{3})^{n+1}$, and $\inf F f_{n+1} = 0$.

(4) The series $\sum_{n=1}^{\infty} g_n$ converges uniformly to a continuous function $g$.

(5) Since $f_n = f - \sum_{r=1}^{n} g_r$, the series $\sum_{n=1}^{\infty} g_n|_F$ converges uniformly to $f$.

Therefore $g|_F = f$.

(6) Note that with this construction $\inf_X g = 0$ and $\sup_X g = 1$.

Now consider the case when $f$ is not bounded.

(1) Suppose that $f$ is neither bounded from below nor bounded from above.

Let $h$ be a homeomorphism from $(-\infty, \infty)$ to $(0, 1)$. Then the range of
1.11. URYSOHN LEMMA

Let $f_1 = h \circ f$ be a subset of $(0, 1)$, therefore it can be extended as in the previous case to a continuous function $g_1$ such that $\inf_{x \in X} g_1(x) = \inf_{x \in F} f_1(x) = 0$ and $\sup_{x \in X} g_1(x) = \sup_{x \in F} f_1(x) = 1$.

If the range of $g_1$ includes neither 0 nor 1 then $g = h^{-1} \circ g_1$ will be the desired function.

It may happen that the range of $g_1$ includes either 0 or 1. In this case let $C = g_1^{-1}((0, 1))$. Note that $C \cap F = \emptyset$. By Urysohn lemma, there is a continuous function $k : X \to [0, 1]$ such that $k|_C = 0$ and $k|_F = 1$. Let $g_2 = kg_1 + (1 - k)\frac{1}{2}$. Then $g_2|_F = g_1|_F$ and the range of $g_2$ is a subset of $(0, 1)$ ($g_2(x)$ is a certain convex combination of $g_1(x)$ and $\frac{1}{2}$). Then $g = h^{-1} \circ g_2$ will be the desired function.

(2) If $f$ is bounded from below then similarly to the previous case we can use a homeomorphism $h : (a, \infty) \to [0, 1)$, and we let $C = g_1^{-1}(\{1\})$.

The case when $f$ is bounded from above is similar.

Problems.

1.11.5. A normal space is completely regular. So: normal $\Rightarrow$ completely regular $\Rightarrow$ regular.
In other words: $T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3$.

1.11.6. Show that the Tietze extension theorem implies the Urysohn lemma.

1.11.7. The Tietze extension theorem is not true without the condition that the set $F$ is closed.

1.11.8. Show that the Tietze extension theorem can be extended to maps to the space $\prod_{i \in I} \mathbb{R}$ where $\mathbb{R}$ has the Euclidean topology.

1.11.9. Let $X$ be a normal space and $F$ be a closed subset of $X$. Then any continuous map $f : F \to S^n$ can be extended to an open set containing $F$.

Hint: Use 1.11.8

1.11.10. * Show that if $X$ is Hausdorff and $Y$ is a retract of $X$ then $Y$ is closed in $X$.

Hint: Use nets.

1.11.11. Prove the following version of Urysohn lemma, as stated in [Rud86].
Suppose that $X$ is a locally compact Hausdorff space, $V$ is open in $X$, $K \subset V$, and $K$ is compact. Then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ for $x \in K$ and $\text{supp}(f) \subset V$, where $\text{supp}(f)$ is the closure of the set $\{x \in X \mid f(x) \neq 0\}$, called the support of $f$.

Hint: Use 1.10.29 and 1.8.22

1.11.12 (Niemyczki space). * Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ be the upper half-plane. Equip $\mathbb{H}$ with the topology generated by the Euclidean open disks (i.e. open balls) in $K = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, together with sets of the form $\{p\} \cup D$ where $p$ is a point on the line $L = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$ and $D$ is an open disk in $K$ tangent to $L$ at $p$. This is called the Niemyczki space.

(1) Check that this is a topological space.
(2) What is the subspace topology on $L$?
(3) What are the closed sets in $H$?
(4) Show that $H$ is Hausdorff.
(5) Show that $H$ is regular.
(6) Show that $H$ is not normal.

*Hint:* Any real function on $L$ is continuous. The cardinality of the set of such function is $c^c$. A real function on $H$ is continuous if and only if it is continuous on the dense subset of points with rational coordinates, so the cardinality of the set of such functions is at most $c^{\aleph_0}$. Since $c^c > c^{\aleph_0}$, the space $H$ cannot be normal, by Tietze Extension Theorem.

**Further readings**

**Metrizability.** A space is said to be metrizable if its topology can be generated by a metric.

**Theorem 1.11.13 (Urysohn Metrizability Theorem).** A regular space with a countable basis is metrizable.

The proof uses the Urysohn lemma [Mun00].
Guide for Further Reading in General Topology

The book by Kelley [Kel55] has been a classic and a standard reference although it was published in 1955. Its presentation is rather abstract. The book contains no figure!

Munkres’ book [Mun00] is famous and is probably the best textbook at present. The treatment there is somewhat more modern than that in Kelley’s book, with many examples, figures and exercises. It also has a section on Algebraic Topology.

Hocking and Young’s book [HY61] contains many deep and difficult results. This book together with Kelley’s and Munkres’ books contain many topics not discussed in our lectures.

The more recent book by Roseman [Ros99] works mostly in $\mathbb{R}^n$, is more down-to-earth, and contains many new topics such as knots and manifolds. The new textbook [AF08] contains many interesting applications of topology.

Some other good books on General Topology are the books by Cain [Cai94], Viro and colleagues [VINK08].

If you want to have some ideas about current research in General Topology you can visit the website of Topology Atlas [Atl], or you can browse the journal Topology and Its Applications, available on the web.

For General Topology as a service to Analysis, [KF75] is an excellent textbook.
CHAPTER 2

Geometric Topology

2.1. Quotient Space

Suppose that $X$ is a topological space, $Y$ is a set, and $f : X \rightarrow Y$ is a map. We want to find a topology on $Y$ such that $f$ becomes a continuous map. Define $V \subset Y$ to be open in $Y$ if $f^{-1}(V)$ is open in $X$. This is a topology on $Y$, called the \textit{quotient topology} (tôpô thương) generated by $f$, denoted by $Y/f$. This is the finest topology on $Y$ such that $f$ is continuous, see 1.3.13.

Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$. Let $p : X \rightarrow X/\sim$ be the projection $x \mapsto [x]$. The set $X/\sim$ with the quotient topology generated by $p$ is called the \textit{quotient space} of $X$ by $\sim$. We can think of the quotient space $X/\sim$ as the space obtained from $X$ by gluing all the points which are equivalent to each other into one point.

If $A \subset X$ then there is an equivalence relation on $X$: $x \sim x$ if $x \notin A$, and $x \sim y$ if $x, y \in A$. The quotient space $X/\sim$ is also denoted by $X/A$. We can think of $X/A$ as the space obtained from $X$ by collapsing the whole subspace $A$ into one point.

From the definition of quotient topology we get a characterization of continuous maps on quotient spaces:

\textbf{Theorem 2.1.1.} Suppose that $f : X \rightarrow Y$ and $Y$ has the quotient topology corresponding to $f$. Let $Z$ be a topological space. Then $g : Y \rightarrow Z$ is continuous if and only if $g \circ f$ is continuous.

\[ X \xrightarrow{f} Y \xrightarrow{g} Z \]

The following result will provide us a tool to identify some quotient spaces:

\textbf{Theorem 2.1.2.} Suppose that $X$ is compact and $\sim$ is an equivalence relation on $X$. Suppose that $Y$ is Hausdorff, and $f : X \rightarrow Y$ is continuous and onto. Suppose that $f(x_1) = f(x_2)$ if and only if $x_1 \sim x_2$. Then $f$ induces a homeomorphism from $X/\sim$ to $Y$.

More concisely, define $h : X/\sim \rightarrow Y$ by $h([x]) = f(x)$, then $f = h \circ p$ and $h$ is a homeomorphism.
Example 2.1.3 (Gluing the two end-points of a line segment gives a circle). More precisely $[0, 1]/0 \sim 1$ is homeomorphic to $S^1$:

$$
\begin{array}{c}
[0, 1] \\
\downarrow^f \\
S^1
\end{array}
$$

Here $f$ is the map $t \mapsto (\cos(2\pi t), \sin(2\pi t))$.

Example 2.1.4 (Gluing a pair of opposite edges of a square gives a cylinder). Let $X = [0, 1] \times [0, 1]/ \sim$ where $(0, t) \sim (1, t)$ for all $0 \leq t \leq 1$. Then $X$ is homeomorphic to the cylinder $[0, 1] \times S^1$. The homeomorphism is induced by the map $(s, t) \mapsto (s, \cos(2\pi t), \sin(2\pi t))$.

Example 2.1.5 (Gluing opposite edges of a square gives a torus). Let $X = [0, 1] \times [0, 1]/ \sim$ where $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, t)$ for all $0 \leq s, t \leq 1$, then $X$ is homeomorphic to the torus (mặt xuyến) $T^2 = S^1 \times S^1$.

The torus $T^2$ is homeomorphic to a subspace of $\mathbb{R}^3$ (we say that the torus can be embedded in $\mathbb{R}^3$). The subspace is the surface of revolution obtained by revolving a circle around a line not intersecting it.

Suppose that the circle is on the $Oyz$-plane, the center is on the $y$-axis and the axis for the rotation is the $z$-axis. Let $a$ be the distance from the center of the circle to the $z$-axis, $b$ be the radius of the circle ($a > b$). Let $S$ be the surface of revolution,

---

The plural form of the word torus is tori.
then the embedding is given by
\[ [0, 2\pi] \times [0, 2\pi] \xrightarrow{f} T^2 = ([0, 2\pi]/0 \sim 2\pi) \times ([0, 2\pi]/0 \sim 2\pi) \]
where \( f(s, t) = ((a + b \cos(s)) \cos(t), (a + b \cos(s)) \sin(t), b \sin(s)) \).

We also obtain an implicit equation for this surface:
\[ (\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2. \]

**Example 2.1.6 (Gluing the boundary circle of a disk together gives a sphere).**
More precisely \( D^2 / \partial D^2 \) is homeomorphic to \( S^2 \). We only need to construct a continuous map from \( D^2 \) onto \( S^2 \) such that after quotient out by the boundary \( \partial D^2 \) it becomes injective, see Figure 2.1.6.

**Example 2.1.7 (The Mobius band).** Gluing a pair of opposite edges of a square in opposite directions gives the Mobius band (dải, lá Mobius). More precisely the Mobius band is \( X = [0, 1] \times [0, 1] / \sim \) where \( (0, t) \sim (1, 1-t) \) for all \( 0 \leq t \leq 1 \).

The Mobius band could be embedded in \( \mathbb{R}^3 \). It is homeomorphic to a subspace of \( \mathbb{R}^3 \) obtained by rotating a straight segment around the \( z \)-axis while also turning that segment “up side down”. The embedding can be induced by the map \( (s, t) \mapsto ((a + t \cos(s/2)) \cos(s), (a + t \cos(s/2)) \sin(s), t \sin(s/2)) \), with \( 0 \leq s \leq 2\pi \) and \( -1 \leq t \leq 1 \).

**Example 2.1.8 (The projective plane).** Identifying opposite points on the boundary of a disk (they are called antipodal points) we get a topological space called the projective plane (mặt phẳng xạ ảnh) \( \mathbb{R}P^2 \).
The real projective plane cannot be embedded in \( \mathbb{R}^3 \). It can be embedded in \( \mathbb{R}^4 \).

**Example 2.1.9 (The projective space).** More generally, identifying antipodal points of \( S^n \), or homeomorphically, identifying antipodal boundary points of \( D^n \) gives us the \textit{projective space} (không gian xạ ảnh) \( \mathbb{R}P^n \).

**Example 2.1.10 (The Klein bottle).** Identifying one pair of opposite edges of a square and the other pair in opposite directions gives a topological space called the \textit{Klein bottle}. More precisely it is \([0,1] \times [0,1] / \sim\) with \((0, t) \sim (1, t)\) and \((s, 0) \sim (1 - s, 1)\).
This space cannot be embedded in $\mathbb{R}^3$, but it can be \textit{immersed} in $\mathbb{R}^3$. An \textit{immersion} (phép nhúng chìm) is a local embedding. More concisely, $f : X \to Y$ is an immersion if each point in $X$ has a neighborhood $U$ such that $f|_U : U \to f(U)$ is a homeomorphism. Intuitively, an \textit{immersion allows self-intersection} (tự cắt).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{klein_bottle_immersed.png}
\caption{The Klein bottle immersed in $\mathbb{R}^3$.}
\end{figure}

**Problems.**

2.1.11. Describe the space $[0, 1]/\frac{1}{2} \sim 1$.

2.1.12. Show that $\mathbb{RP}^2$ is homeomorphic to the sphere with antipodal points identified, that is, $S^2/\sim = \mathbb{R}P^2$.

2.1.13. On the Euclidean $\mathbb{R}$ define $x \sim y$ if $x - y \in \mathbb{Z}$. Show that $\mathbb{R}/\sim$ is homeomorphic to $S^1$.

The space $\mathbb{R}/\sim$ is also described as “$\mathbb{R}$ quotient by the action of the group $\mathbb{Z}$”.

2.1.14. On the Euclidean $\mathbb{R}^2$, define $(x_1, y_1) \sim (x_2, y_2)$ if $(x_1 - x_2, y_1 - y_2) \in \mathbb{Z} \times \mathbb{Z}$. Show that $\mathbb{R}^2/\sim$ is homeomorphic to $T^2$.

2.1.15. Show that the following spaces are homeomorphic (one of them is the Klein bottle).

\begin{center}
\begin{tabular}{ccc}
\textbf{a} & \textbf{b} & \textbf{a} \\
\textbf{b} & \textbf{a} & \textbf{b} \\
\end{tabular}
\end{center}

2.1.16. Describe the space that is the sphere $S^2$ quotient by its equator $S^1$.

2.1.17. If $X$ is connected then $X/\sim$ is connected.

2.1.18. The one-point compactification of the open Mobius band (the Mobius band without the boundary circle) is the projective space $\mathbb{RP}^2$.

2.1.19. Show that $\mathbb{RP}^1$ is homeomorphic to $S^1$.

2.1.20. Show that $\mathbb{RP}^n$ is a compactification of $\mathbb{R}^n$.

2.1.21. In order for the quotient space $X/\sim$ to be Hausdorff, a necessary condition is that each equivalence class $[x]$ must be a closed subset of $X$. Is this condition sufficient?
2.2. Topological Manifolds

The idea of a manifold is quite simple and natural. We are living on the surface of the Earth, which is of course a sphere. Now, if we don’t travel far and only stay around our small familiar neighborhood, then we don’t recognize the curvature of the surface and to us it is undistinguishable from a plane. This is the reason why in ancient time people thought that the Earth was flat. So we see that globally a sphere is different from a plane, but locally is same. The idea of manifold, put forth for the first time by Bernard Riemann, is this: 

A space is a manifold if locally it is the same as a Euclidean space.

Manifold is a central object in modern mathematics.

Definition 2.2.1. A topological manifold (da tap tôpô) of dimension $n$ is a topological space each point of which has a neighborhood homeomorphic to the Euclidean space $\mathbb{R}^n$.

Remark 2.2.2. In this section we assume $\mathbb{R}^n$ has the Euclidean topology unless we mention otherwise.

Proposition 2.2.3. An equivalent definition of manifold is: A manifold of dimension $n$ is a space such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

We can think of a manifold as a space which could be covered by a collection of open sets each of which homeomorphic to $\mathbb{R}^n$.

Remark 2.2.4. By Invariance of dimension $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic unless $m = n$, therefore a manifold has a unique dimension.

Example 2.2.5. Any open subset of $\mathbb{R}^n$ is a manifold of dimension $n$.

Example 2.2.6. If $f : \mathbb{R} \to \mathbb{R}$ is continuous then the graph of $f$ is a one-dimensional manifold. More generally, let $f : D \to \mathbb{R}$ be a continuous function where $D \subset \mathbb{R}^n$ is an open set. Then the graph of $f$, the set $\{(x, f(x)) \mid x \in D\}$ as a subspace of $\mathbb{R}^{n+1}$ is an $n$-dimensional manifold.
Example 2.2.7. The sphere $S^n$ is an $n$-dimensional manifold.

One way to show this is to cover $S^n$ with two neighborhoods $S^n \setminus \{(0,0,\ldots,0,1)\}$ and $S^n \setminus \{(0,0,\ldots,0,-1)\}$. Each of these neighborhoods is homeomorphic to $\mathbb{R}^n$ via stereographic projections.

Another way is covering $S^n$ by hemispheres $\{(x_1,x_2,\ldots,x_{n+1}) \in S^n \mid x_i > 0\}$ and $\{(x_1,x_2,\ldots,x_{n+1}) \in S^n \mid x_i < 0\}$, $1 \leq i \leq n+1$.

Example 2.2.8. The torus is a two-dimensional manifold.

Let us consider the torus as the quotient space of the square $[0,1]^2$ by identifying opposite edges. Each point has a neighborhood homeomorphic to an open disk, as can be seen easily in the following figure, though explicit description would be time consuming.

We can also view the torus as a surface in $\mathbb{R}^3$, given by the equation $(\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2$. As such it can be covered by the open subsets corresponding to $z > 0, z < 0, x^2 + y^2 < a^2, x^2 + y^2 > a^2$.

Remark 2.2.9. The interval $[0,1]$ is not a manifold. It is a manifold with boundary. We will not give a precise definition of manifold with boundary here.

Problems.

2.2.10. Show that if two spaces are homeomorphic and one space is an $n$-dimensional manifold then the other is also an $n$-dimensional manifold.

2.2.11. Show that $\mathbb{R}P^n$ is an $n$-dimensional topological manifold.

Further readings

Bernard Riemann proposed the idea of manifold in his Habilitation dissertation. A translation of this article is available in [Spi99].
Two conditions are often added to the definition of a manifold: it is Hausdorff, and it has a countable basis. The first condition is useful for doing Analysis on manifolds, and the second condition guarantees the existence of Partition of Unity.

**Theorem 2.2.12 (Partition of Unity).** Let \( U \) be an open cover of a manifold \( M \). Then there is a collection \( F \) of continuous real functions \( f : M \to [0,1] \) such that

1. For each \( f \in F \), there is \( V \in U \) such that \( \text{supp}(f) \subset V \).
2. For each \( x \in M \) there is a neighborhood of \( x \) such that there are only finitely many \( f \in F \) which is non-zero on that neighborhood.
3. For each \( x \in M \), \( \sum_{f \in F} f(x) = 1 \)

A Partition of Unity allows us to extend some local properties to global ones, by “patching” neighborhoods. It is needed for such important results as the existence of a Riemannian metric on a manifold in Differential Geometry, the definition of integration on manifold in Theory of Differential Forms. It is also used in the proof of the Riesz Representation Theorem in Measure Theory [Rud86].

2.2.13. Check that \( \mathbb{R}^n \) has a countable basis.

*Hint:* The set of all balls with rational radius whose center has rational coordinates forms a basis for the Euclidean topology of \( \mathbb{R}^n \).

2.2.14. Any subset of \( \mathbb{R}^n \) is Hausdorff and has a countable basis.

With the above additional assumptions we can show:

2.2.15. A manifold is locally compact.

2.2.16. A manifold is a regular space.

*Hint:* By [1.10.22]

By the Urysohn Metrizability Theorem [1.11.13] we have:

2.2.17. A manifold is metrizable.
2.3. Classification of Compact Surfaces

In this section by a surface (mặt) we mean a two-dimensional manifold, with or without boundary. Often we further assume that it is connected and compact.

We already know some surfaces: the sphere, the torus, the projective plane, the Klein bottle. Now we will list all possible compact surfaces.

Connected sum. Let \( S \) and \( T \) be two surfaces. From each surface deletes an open disk, obtaining a surface with a circle boundary. Glue the two surfaces along the two boundary circles. The resulting surface is called the connected sum (tổng trực tiếp) of the two surfaces, denoted by \( S \# T \).

The connected sum does not depend on the choices of the disks.

Example 2.3.1. If \( S \) is any surface then \( S \# S^2 = S \).

Classification.

Theorem 2.3.2 (Classification of compact boundaryless surfaces). A connected compact without boundary surface is homeomorphic to either the sphere, or a connected sum of tori, or a connected sum of projective planes.

We denote by \( T_g \) the connected sum of \( g \) tori, and by \( M_g \) the connected sum of \( g \) projective planes. The number \( g \) is called the genus (giống) of the surface.

The sphere and the surfaces \( T_g \) are orientable (định hướng được) surfaces, while the surfaces \( M_g \) are non-orientable (không định hướng được) surfaces. We will not give a precise definition of orientability here.

Notice that at this stage we have not yet been able to prove that those surfaces are distinct.
**Triangulation.** A *triangulation* (phép phân chia tam giác) of a surface is a homeomorphism from the surface to a union of finitely many triangles, with a requirement that two triangles are either disjoint, or have one common edge, or have one common vertex.

Intuitively, the surface is expressed as a proper union of triangles.

---

**Figure 2.3.2.** A triangulation of the sphere.

**Figure 2.3.3.** A triangulation of the torus.

**Figure 2.3.4.** Another triangulation of the torus.

It is known that any surface can be triangulated, this was proved in the 1920’s.
Euler Characteristics. The Euler Characteristics (đặc trưng Euler) $\chi(S)$ of a triangulated surface $S$ is the number $V$ of vertices minus the number $E$ of edges plus the number $F$ of triangles (faces):

$$\chi(S) = V - E + F$$

Theorem 2.3.3. The Euler Characteristics with respect to two triangulations of the same surface are equal.

By this theorem the Euler Characteristics of a surface is defined and does not depend on the choice of triangulation. Since the Euler Characteristics does not change under homeomorphisms, just like the number of connected components, it is said to be a topological invariant (bất biến tôpô). If two surfaces have different Euler Characteristics, then they are not homeomorphic.

Example 2.3.4. By Theorem 2.3.3 we have $\chi(S^2) = 2$.

A consequence is the famous formula of Leonhard Euler: For any convex polyhedron, $V - E + F = 2$.

From any triangulation of the torus, we get $\chi(T^2) = 0$. For the projective plane, $\chi(\mathbb{R}P^2) = 1$.

As a consequence, the sphere, the torus, and the projective plane are not homeomorphic to each other: they are different surfaces.

Problems.

2.3.5. (a) Show that $\mathbb{R}P^2$ with an open disk removed is the Mobius band.

(b) Show that $T^2 \# \mathbb{R}P^2 = K \# \mathbb{R}P^2$, where $K$ is the Klein bottle.

2.3.6. Show that gluing two Mobius band along their boundaries gives the Klein bottle. In other words, $\mathbb{R}P^2 \# \mathbb{R}P^2 = K$.

2.3.7 (Surfaces are homogeneous). A space is homogeneous (đồng nhất) if given two points there exists a homeomorphism from the space to itself bringing one point to the other point.
(1) Show that the sphere \( S^2 \) is homogeneous.
(2) Show that the torus \( T^2 \) is homogeneous.

It is known that any manifold is homogeneous, see 3.7.4.

2.3.8. (a) Show that \( T_g \# T_h = T_{g+h} \).
   (b) Show that \( M_g \# M_h = M_{g+h} \).
   (c) What is \( M_g \# T_h \)?

2.3.9. Show that \( \chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2 \).
   Hint: Deleting an open disk is the same as deleting the interior of a triangle.

2.3.10. * Compute the Euler Characteristics of all connected compact without boundary surfaces.
   Deduce that the orientable surfaces \( S^2 \) and \( T_g \), for different \( g \), are distinct, meaning not homeomorphic to each other. Similarly the non-orientable surfaces \( M_g \) are all distinct.

Further readings

Proof of the Classification Theorem. Let \( S \) be a triangulated surface. Cut \( S \) along the triangles. Label the edges by alphabet characters and mark the orientations of each edge. In this way each edge will appear twice on two different triangles.

Take one triangle. Pick a second triangle which has one common edge with the first one, then glue the two along the common edge following the orientation of the edge. Continue this gluing process in such a way that at every step the resulting polygon is planar. This is possible if at each stage the gluing is done in such a way that there is one edge of the polygon such that the entire polygon is on one of its side. The last polygon \( P \) is called a fundamental polygon of the surface.

The boundary of the fundamental polygon consists of labeled and oriented edges. Choose one edge as the initial one then follow the edge of the polygon in a predetermined direction. This way we associate each polygon with a word \( w \).

We consider two words equivalent if they give rise to homeomorphic surfaces.

Example 2.3.11. A fundamental polygon of the sphere and its associated word.

In the reverse direction, the surface can be reconstructed from an associated word. We consider two words equivalent if they give rise to homeomorphic surfaces. In order to find all possible surfaces we will find all possible associated words up to equivalence.

Theorem 2.3.12. An associated word to a connected compact without boundary surface is equivalent to a word of the forms:

1. \( aa^{-1} \),
2. \( a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \),
3. \( c_1^2 c_2^2 \cdots c_g^2 \).

A direct consequence is:

Theorem 2.3.13. A connected compact without boundary surface is homeomorphic to a surface determined by a word of the forms:

1. \( aa^{-1} \),
2. \( a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \),
3. \( c_1^2 c_2^2 \cdots c_g^2 \).

This theorem gives the the Classification Theorem.
Let \( w \) be a word of a fundamental polygon.

**Lemma 2.3.14.** A pair of the form \( a a^{-1} \) in \( w \) can be deleted, meaning that this action will give an equivalent word corresponding to a homeomorphic surface.

**Proof.** If \( w \) is not \( a a^{-1} \) then it can be reduced as illustrated in the figure.

\[
\begin{array}{ccc}
\text{a} & \text{a}^{-1} \\
\end{array}
\]

**Figure 2.3.6. Lemma 2.3.14.**

**Lemma 2.3.15.** The word \( w \) is equivalent to a word whose all of the vertices of the associated polygon is identified to a single point on the associated surface (\( w \) is "reduced").

**Proof.** When we do the following operation, the number of \( P \) vertices is decreased.

\[
\begin{array}{ccc}
\text{P} & \text{b} & \text{a} & \text{c} & \text{P} \\
\text{Q} & \text{b} & \text{c} & \text{c} & \text{P} \\
\end{array}
\]

**Figure 2.3.7. Lemma 2.3.15.**

When there is only one \( P \) vertex left, we arrive at the situation in Lemma 2.3.14.

**Lemma 2.3.16.** A word of the form \(-a - a-\) is equivalent to a word of the form \(-aa-\).
Lemma 2.3.17. Suppose that $w$ is reduced. Assume that $w$ has the form $-aaa^{-1} -$ where $a$ is a non-empty word. Then there is a letter $b$ such that $b$ is in $a$ but the other $b$ or $b^{-1}$ is not.

**Proof.** If all letters in $a$ appear in pairs then the vertices in the part of the polygon associated to $a$ are identified only with themselves, and are not identified with a vertex outside of that part. This contradicts the assumption that $w$ is reduced. \qed

Lemma 2.3.18. A word of the form $-a - b - a^{-1} - b^{-1} -$ is equivalent to a word of the form $-aba^{-1}b^{-1} -$.

**Lemma 2.3.19.** A word of the form $-aba^{-1}b^{-1} - cc -$ is equivalent to a word of the form $-a^2 - b^2 - c^2 -$.
2.3. CLASSIFICATION OF COMPACT SURFACES

PROOF. Do the operation in the figure, after that we are in a situation where we can apply Lemma 2.3.16 three times.

![Diagram](image)

**Figure 2.3.10. Lemma 2.3.19.**

PROOF OF THEOREM 2.3.12. The proof follows the following steps.

1. Bring \( w \) to the reduced form by using 2.3.15 finitely many times.
2. If \( w \) has the form \(-aa^{-1}-\) then go to 2.1, if not go to 3.

2.1. If \( w \) has the form \( aa^{-1} \) then stop, if not go to 2.2.

2.2. \( w \) has the form \( aa^{-1}a \) where \( a \neq \emptyset \). Repeatedly apply 2.3.14 finitely many times, deleting pairs of the form \( aa^{-1} \) in \( w \) until no such pair is left or \( w \) has the form \( aa^{-1} \). If no such pair is left go to 3.

3. \( w \) does not have the form \(-aa^{-1}-\). Repeatedly apply 2.3.16 finitely many times until \( w \) no longer has the form \(-aaa-\) where \( a \neq \emptyset \). Note that if we apply 2.3.16 then some pairs of the form \(-aaa-\) with \( a \neq \emptyset \) could become a pair of the form \(-a-a^{-1}-\), but a pair of the form \(-aa-\) will not be changed. Therefore 2.3.16 could be used finitely many times until there is no pair \(-aaa-\) with \( a = \emptyset \) left.

Also it is crucial from the proof of 2.3.16 that this step will not undo the steps before it.

4. If there is no pair of the form \(-aaa^{-1}\) where \( a \neq \emptyset \), then stop: \( w \) has the form \( a_1^2a_2^2 \cdots a_g^2 \).

5. \( w \) has the form \(-aaa^{-1}\) where \( a \neq \emptyset \). By 2.3.17 \( w \) must have the form \(-a-b-a^{-1}b^{-1}-\), since after Step 3 there could be no \(-b-a^{-1}b-\).

6. Repeatedly apply 2.3.18 finitely many times until \( w \) no longer has the form \(-aab\beta a^{-1}\gamma b^{-1}-\) where at least one of \( a, \beta, \) or \( \gamma \) is non-empty.

7. If \( w \) is not of the form \(-aa-\) then stop: \( w \) has the form \( a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1} \).

8. \( w \) has the form \(-aba^{-1}b^{-1}cc-\). Use 2.3.19 finitely many times to transform \( w \) to the form \( a_1^2a_2^2 \cdots a_g^2 \). □
2.4. Homotopy

**Homotopy of paths.** Recall that a *path* (đường đi) in a space $X$ is a continuous map $\alpha$ from the Euclidean interval $[0, 1]$ to $X$. The point $\alpha(0)$ is called the initial end point, and $\alpha(1)$ is called the final end point. In this section for simplicity of presentation we assume the domain of a path is the Euclidean interval $[0, 1]$ instead of any Euclidean closed interval as before.

Let $\alpha$ and $\beta$ be two paths from $a$ to $b$ in $X$. A *homotopy* (phép đồng luân) from $\alpha$ to $\beta$ is a family of maps $F_t : X \to X$, $t \in [0, 1]$, such that the map $(t, s) \mapsto F_t(s)$ is continuous, $F_0 = \alpha$, $F_1 = \beta$, and for each $t$ the path $F_t$ goes from $a$ to $b$.

If there is a homotopy from $\alpha$ to $\beta$ we say that $\alpha$ is *homotopic* (đồng luân) to $\beta$, sometimes written $\alpha \sim \beta$.

![Figure 2.4.1](image)

**Figure 2.4.1.** We can think of $\alpha$ being homotopic to $\beta$ as there is a way (a homotopy) to continuously brings $\alpha$ to $\beta$, similar to a motion picture.

**Example 2.4.1.** In a normed space any two paths $\alpha$ and $\beta$ with the same initial points and end points are homotopic, via the homotopy $(1 - t)\alpha + t\beta$. This is also true for any convex subset of a normed space.

**Proposition 2.4.2.** Homotopic relation on the set of all paths from $a$ to $b$ is an equivalence relation.

**Proof.** If $\alpha$ is homotopic to $\beta$ via a homotopy $F$ then we can easily find a homotopy from $\beta$ to $\alpha$, for instance $G_t = F_{1-t}$.

We check that if $\alpha$ is homotopic to $\beta$ via a homotopy $F$ and $\beta$ is homotopic to $\gamma$ via a homotopy $G$ then $\alpha$ is homotopic to $\gamma$. Let

$$H_t = \begin{cases} 
  F_{2t}, & 0 \leq t \leq \frac{1}{2} \\
  G_{2t-1}, & \frac{1}{2} \leq t \leq 1
  \end{cases}$$

Note that continuity of a map is not the same as continuity with respect to each variable (see 1.9.8). However by an argument similar to 1.4.13 we can check directly that the map $H$ is continuous. It is a homotopy from $\alpha$ to $\gamma$. $\square$

A *loop* (vòng) or a *closed path* (đường đi đóng) at $a \in X$ is a path whose initial point and end point are $a$. In other words it is a continuous map $\alpha : [0, 1] \to X$ such that $\alpha(0) = \alpha(1) = a$. The *constant loop* is the loop $\alpha(t) = a$ for all $t \in [0, 1]$. 

---

**Note:** This text is a translation of a page from a mathematics textbook, focusing on geometric topology, with specific emphasis on homotopy theory. The content covers fundamental concepts in this area, including the definition of paths, homotopies, and related theorems and examples. The text is presented in a clear and logical manner, suitable for an introductory course in topology. The inclusion of figures and examples aids in understanding the abstract concepts discussed.
A space is said to be simply connected (đơn liên) if it is path-connected and any loop is homotopic to a constant loop.

**Example 2.4.3.** As in a previous example, any convex subset of a normed space is simply connected.

**Deformation retract.** Let $X$ be a space, and let $A$ be a subspace of $X$. A deformation retraction (phép rút biến dạng) from $X$ to $A$ is a family of maps $F_t : X \to X$, $t \in [0, 1]$, such that the map $(t, x) \mapsto F_t(x)$ is continuous, $F_0 = \text{id}_X$, $F_t|A = \text{id}_A$, and $F_1(X) = A$. If there is such a deformation retraction we say that $A$ is a deformation retract (rút biến dạng) of $X$. In this case for each point $x \in X \setminus A$ there is a path (namely $F_t(x)$) that goes from $x$ to a point in $A$ (namely $F_1(x)$).

**Example 2.4.4.** A normed space minus a point has a deformation retraction to a sphere. Indeed a normed space minus the origin has a deformation retraction $F_t(x) = (1 - t)x + t\frac{x}{||x||}$ to the unit sphere at the origin.

**Example 2.4.5.** An annulus $S^1 \times [0, 1]$ has a deformation retract to one of its circle boundary $S^1 \times \{0\}$.

**Homotopy of maps.** Now we generalize the notion of homotopy of paths to homotopy of maps between two spaces. Let $f$ and $g$ be continuous maps from $X$ to $Y$. We say that $f$ is homotopic to $g$ if there is a family of maps $F_t : X \to Y$, $t \in [0, 1]$, such that continuous map, called a homotopy, $F : X \times [0, 1] \to Y$, often written $F_t(x)$, such that $F_0 = f$ and $F_1 = g$.

Clearly, if $f$ is homotopic to $g$ then $g$ is homotopic to $f$.

**Remark 2.4.6.** Homotopy of paths is a special case of homotopy of maps, with the further requirement that the homotopy fixes the initial point and the end point.

**Definition 2.4.7.** The space $X$ is said to be homotopic to the space $Y$ if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity map $\text{id}_X$ and $f \circ g$ is homotopic to the identity map $\text{id}_Y$.

**Proposition 2.4.8.** If a space $X$ has a deformation retraction to a subspace $A$ then $X$ is homotopic to $A$.

**Proof.** Suppose that $F_t$ is a deformation retraction from $X$ to $A$. Consider $F_1 : X \to A$ and the inclusion map $g : A \to X$, $g(x) = x$. Then $\text{id}_X$ is homotopic to $g \circ F_t$ via $F_t$, while $F_t \circ g = \text{id}_A$. □

**Example 2.4.9.** The annulus and the Mobius band are both homotopic to the circle, but are not homeomorphic to it.

**Example 2.4.10.** The letter $A$ is homotopic to the letter $O$, as subspaces of the Euclidean plane.

A space which is homotopic to a space containing only one point is called a contractible space (thắt được).
Example 2.4.11. A ball in a normed space is contractible.

Example 2.4.12. A subset $A$ of $\mathbb{R}^n$ is called star-shaped if there is a point $x_0 \in A$ such that for any $x \in A$ the straight segment from $x$ to $x_0$ is contained in $A$. Since $A$ has a deformation retract to $x_0$, it is contractible.

Theorem 2.4.13. The circle is not contractible.

This important result is often the beginning of either Algebraic Topology or Differential Topology. It has many nice consequences.

The fundamental group. Let $\alpha$ be a path from $a$ to $b$. Then the inverse path of $\alpha$ is defined to be the path $\alpha^{-1}(t) = \alpha(1 - t)$ from $b$ to $a$.

Let $\alpha$ be a path from $a$ to $b$, and $\beta$ be a path from $b$ to $c$, then the composition (hop) of $\alpha$ with $\beta$ is defined to be the path

$$\gamma(t) = \begin{cases} 
\alpha(2t), & 0 \leq t \leq \frac{1}{2} \\
\beta(2t - 1), & \frac{1}{2} \leq t \leq 1.
\end{cases}$$

The path $\gamma$ is often denoted as $\alpha \cdot \beta$. By 1.4.13, $\alpha \cdot \beta$ is continuous.

Proposition 2.4.14. If $\alpha \sim \alpha_1$ and $\beta \sim \beta_1$ then $\alpha \cdot \beta \sim \alpha_1 \cdot \beta_1$. Therefore we can define $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$.

Proof. Let $F$ be the first homotopy and $G$ be the second homotopy. Consider $H_t = F_t \cdot G_t$.

Proposition 2.4.15. If $\alpha$ is a path from $a$ to $b$ then $\alpha \cdot \alpha^{-1}$ is homotopic to the constant loop at $a$.

Proof. Our homotopy from $\alpha \cdot \alpha^{-1}$ to the constant loop at $a$ can be described as follows. At a fixed $t$, the loop $F_t$ starts at time 0 at $a$, goes along $\alpha$ but at twice the speed of $\alpha$, until time $\frac{1}{2} - \frac{t}{2}$, stays there until time $\frac{1}{2} + \frac{t}{2}$, then catches the inverse path $\alpha^{-1}$ at twice its speed to come back to $a$.

More precisely,

$$F_t(s) = \begin{cases} 
\alpha(2s), & 0 \leq s \leq \frac{1}{2} - \frac{t}{2} \\
\alpha\left(\frac{1}{2} - \frac{t}{2}\right), & \frac{1}{2} - \frac{t}{2} \leq s \leq \frac{1}{2} + \frac{t}{2} \\
\alpha^{-1}(2s), & \frac{1}{2} + \frac{t}{2} \leq s \leq 1.
\end{cases}$$

Theorem 2.4.16. The set of all homotopy classes of loops of $X$ at a point $x_0$ is a group under the above composition.

This group is called the fundamental group (nhóm cơ bản) of $X$ at $x_0$, denoted by $\pi_1(X, x_0)$. The point $x_0$ is called the base point.

Example 2.4.17. If $X$ is a convex subset of a normed space then $\pi_1(X, x_0)$ is trivial.
2.4. HOMOTOPY

The dependence of the fundamental group on the base point is explained in the following proposition.

**Proposition 2.4.18.** Let \( \gamma \) be a path from \( x_0 \) to \( y_0 \). Then the map

\[
\gamma^* : \pi_1(X, x_0) \to \pi_1(X, y_0) \quad [f] \mapsto [\gamma^{-1} \cdot f \cdot \gamma]
\]

is an isomorphism.

Thus for a path-connected space the choice of the base point is not important for the fundamental group.

**Problems.**

2.4.19. Show that the Mobius band has a deformation retract to a circle.

2.4.20. Show that if the space \( X \) is homotopic to the space \( Y \) and \( Y \) is homotopic to the space \( Z \) then \( X \) is homotopic to \( Z \).

2.4.21. If two spaces are homeomorphic then they are homotopic.

2.4.22. Show that the homotopy type of the Euclidean plane with a point removed does not depend on the choice of the point.

2.4.23. Classify the alphabetical letters up to homotopy types.

2.4.24. Show that a contractible space is path-connected.

2.4.25. Let \( X \) be a topological space, and \( Y \) be a subspace of \( X \). We say that \( Y \) is a retract of \( X \) if there is a continuous map \( r : X \to Y \) such that \( r|_Y = \text{id}_Y \). In other words the identity map \( \text{id}_Y \) can be extended to \( X \).

1. Show that if \( Y \) is a retract of \( X \) then any map from \( Y \) to a topological space \( Z \) can be extended to \( X \).

2. Show that a subset consisting of two points cannot be a retract of \( \mathbb{R}^2 \).

Note: This shows that the Tietze extension theorem cannot be automatically generalized to maps to general topological spaces.

**Further readings**

One of the most celebrated achievements in Topology is the resolution of the Poincaré conjecture:

**Theorem 2.4.26 (Poincaré conjecture).** A compact manifold that is homotopic to the sphere is homeomorphic to the sphere.

The proof of this statement is the result of a cummulative effort of many mathematicians, including Stephen Smale (for dimension \( \geq 5 \), early 1960s), Michael Freedman (for dimension 4, early 1980s), and Grigory Perelman (for dimension 3, early 2000s).
CHAPTER 3

Differential Topology

3.1. Smooth Manifolds

Smooth maps on open sets in \( \mathbb{R}^n \). Let \( U \) be an open subset of \( \mathbb{R}^k \). A function \( f : U \to \mathbb{R}^l \) is said to be smooth if all partial derivatives of all orders exist, i.e. \( f \in C^\infty(U) \).

Let \( U \) be an open subset of \( \mathbb{R}^k \) and \( V \) an open subset of \( \mathbb{R}^l \). We say that \( f : U \to V \) is a diffeomorphism if it is bijective and both \( f \) and \( f^{-1} \) are smooth.

Remark 3.1.1. From now on in this chapter we assume that \( \mathbb{R}^n \) has the Euclidean topology.

Smooth manifolds. Let \( X \) be a subset of \( \mathbb{R}^k \). A map \( f : X \to \mathbb{R}^l \) is said to be smooth at \( x \in X \) if it can be extended to a smooth function on a neighborhood of \( x \) in \( \mathbb{R}^k \). Namely there is an open set \( U \subset \mathbb{R}^k \) and a smooth function \( F : U \to \mathbb{R}^l \) such that \( F|_{U \cap X} = f \).

Let \( X \subset \mathbb{R}^k \) and \( Y \subset \mathbb{R}^l \). Then \( f : X \to Y \) is a diffeomorphism if it is bijective and both \( f \) and \( f^{-1} \) are smooth.

Proposition 3.1.2. A set \( M \subset \mathbb{R}^k \) is a smooth manifold of dimension \( m \) if any \( x \in M \) has a neighborhood in \( M \) which is diffeomorphic to an open neighborhood in \( \mathbb{R}^m \).

A pair \( (U, \phi) \) of a neighborhood \( U \subset M \) and a diffeomorphism \( \phi : U \to \mathbb{R}^m \) is called a chart or coordinate system with domain \( U \). The inverse map \( \phi^{-1} \) is called a parametrization.

3.1.3. A diffeomorphism is a homeomorphism. Therefore a smooth manifold is a topological manifold.

Remark 3.1.4. In this part we only study smooth manifolds imbedded in Euclidean spaces. Unless stated otherwise, manifolds mean smooth manifolds.

Remark 3.1.5. In the definition we can replace “an open neighborhood in \( \mathbb{R}^m \)” by “\( \mathbb{R}^m \)”. Thus we can say that a smooth manifold of dimension \( m \) is a subset of a Euclidean space \( \mathbb{R}^k \) that is locally diffeomorphic to \( \mathbb{R}^m \).

Remark 3.1.6. We repeat that by Invariance of dimension, an open set in \( \mathbb{R}^m \) cannot be homeomorphic to an open set in \( \mathbb{R}^n \) if \( m \neq n \), therefore a manifold has a unique dimension.

Example 3.1.7. A manifold of dimension 0 is a discrete set of points.
Example 3.1.8. The graph of a smooth function $y = f(x)$ for $x \in (a, b)$ (a smooth curve) is a 1-dimensional manifold.

Example 3.1.9. The real number line $\mathbb{R}$ is a manifold, but $t \mapsto t^3$ is not a parametrization of this manifold.

Example 3.1.10. On the coordinate plane, the union of the two axes is not a manifold.

Example 3.1.11 (The circle). Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. It is covered by four neighborhoods which are half circles, each corresponds to points $(x, y) \in S^1$ such that $x > 0$, $x < 0$, $y > 0$ and $y < 0$. Each of these neighborhoods is diffeomorphic to $(-1, 1)$. For example consider the projection from $\{(x, y) \in S^1 \mid x > 0\} \to (-1, 1)$ given by $(x, y) \mapsto y$. The map $(x, y) \mapsto y$ is smooth on $\mathbb{R}^2$, so it is smooth on $W_1$. The inverse map $y \mapsto (\sqrt{1-y^2}, y)$ is smooth on $(-1, 1)$. Therefore the projection is a diffeomorphism.

Problems.

3.1.12. * Find a diffeomorphism from an open ball $B(x, r)$ onto $\mathbb{R}^n$.

3.1.13. * The graph of a smooth function $z = f(x, y)$ is a 2-dimensional manifold.

3.1.14. * The sphere $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$ is a differentiable manifold of dimension $n$, covered by the hemispheres.

There is another way to see that $S^n$ as a manifold, by using two stereographic projections, one from the North Pole and one from the South Pole.

3.1.15. Show that the hyperboloid $x^2 + y^2 - z^2 = 1$ is a manifold. Is the surface $x^2 + y^2 - z^2 = 0$ a manifold?

*Hint:* Graph the surfaces.

3.1.16. The torus can be obtained by rotating around the $z$ axis a circle on the $xOz$ plane not intersecting the $z$ axis. Show that the torus is a smooth manifold.

*Hint:* Since $S^1$ can be covered by four neighborhoods, the torus $T^2 = S^1 \times S^1$ can be covered by $4 \times 4$ neighborhoods. It can be shown that the torus has the equation $(\sqrt{x^2+y^2}-b)^2 + z^2 = a^2$ where $0 < a < b$.

3.1.17. (a) Consider the union of the curve $y = \sin \frac{1}{x}, x > 0$ and the segment $\{(0, y) \mid -1 \leq y \leq 1\}$ (the topologist’s sine curve, see also Section 1.5.1). Is it a manifold?

(b) Consider the curve which is the union of the curve $y = x^3 \sin \frac{1}{x}, x \neq 0$ and $y = 0$ when $x = 0$. Is this a manifold?

*Hint:* Consider a neighborhood of a point on the curve on the $y$-axis. Can it be homeomorphic to an open neighborhood in $\mathbb{R}$?

3.1.18. An open subset of a manifold is a manifold.

3.1.19. A connected manifold is path-connected.

*Hint:* See [1.5.22]
3.1.20. * The following is a common smooth function:

\[ f(x) = \begin{cases} 
  e^{-1/x}, & \text{if } x > 0 \\
  0, & \text{if } x \leq 0 
\end{cases} \]

1. The function \( f(x) \) is smooth.  
   \( \text{Hint: } \) Show that \( f^{(n)}(x) = e^{-1/x}p_n(1/x) \) where \( p_n(x) \) is a polynomial.

2. Let \( a < b \) and let \( g(x) = f(x - a)f(b - x) \). Then \( g \) is smooth, \( g(x) \) is positive on \((a, b)\) and is zero everywhere else.

3. Let 
   \[ h(x) = \frac{\int_{-\infty}^{x} g(x) \, dx}{\int_{-\infty}^{\infty} g(x) \, dx} \]
   Then \( h(x) \) is smooth, \( h(x) = 0 \) if \( x \leq a \), \( 0 < h(x) < 1 \) if \( a < x < b \), and \( h(x) = 1 \) if \( x \geq b \).

4. The function 
   \[ k(x) = \frac{f(x - a)}{f(x - a) + f(b - x)} \]
   also has the above properties of \( h(x) \).

5. In \( \mathbb{R}^n \), construct a smooth function whose value is 0 outside of the ball of radius \( b \), 1 inside the ball of radius \( a < b \), and between 0 and 1 in between the two balls.

3.1.21. * Show that any diffeomorphism from \( S^{n-1} \) onto \( S^{n-1} \) can be extended to a diffeomorphism from \( D^n = B'(0, 1) \) onto \( D^n \).

   \( \text{Hint: } \) See 1.4.24 and 3.1.20.
3.2. Tangent Spaces – Derivatives

Tangent spaces and derivatives on open sets in \( \mathbb{R}^n \). We summarize here some results about derivatives of functions defined on open sets in \( \mathbb{R}^n \). See for instance [Spi65] for more details.

Let \( U \) be an open set in \( \mathbb{R}^k \) and \( V \) be an open set in \( \mathbb{R}^l \). Let \( f : U \to V \) be smooth. We define the tangent spaces of \( U \) at \( x \) to be \( TU_x = \mathbb{R}^k \). We define the derivative of \( f \) at \( x \in U \) to be the linear map \( df_x : TU_x \to TV \) such that

\[
    df_x(h) = \lim_{h \to 0} \frac{f(x + th) - f(x)}{t}
\]

Thus \( df_x(h) \) is the directional derivative of \( f \) at \( x \) in the direction of \( h \).

The derivative \( df_x \) is a linear approximation of \( f \) at \( x \).

Because we assumed that all the first order partial derivatives of \( f \) exist and are continuous, the derivative of \( f \) exists. Furthermore in the canonical coordinate system of \( \mathbb{R}^n \) \( df_x \) is represented by an \( l \times k \)-matrix \( J f_x \) with \( 1 \leq i \leq l \) and \( 1 \leq j \leq k \), called the Jacobian of \( f \) at \( x \), and we have \( df_x(h) = J f_x \cdot h \).

Some properties of derivatives:

**Theorem 3.2.1 (The Chain Rule).** Let \( U, V, W \) be open sets in \( \mathbb{R}^k, \mathbb{R}^l, \mathbb{R}^p \) respectively, \( f : U \to V \) and \( g : V \to W \) be smooth maps, and \( y = f(x) \). Then

\[
    d(g \circ f)_x = dg_y \circ df_x.
\]

In other words, the following commutative diagram

\[
\begin{array}{ccc}
    & V & \\
  f & & g \\
  U & \xrightarrow{g \circ f} & W
\end{array}
\]

induces the commutative diagram

\[
\begin{array}{ccc}
    & TV \downarrow & \\
  & \downarrow & \\
  & TU_x & \xrightarrow{d(g \circ f)_x} & TV_y \\
  & \downarrow & \\
  & d g_y \circ df_x & \to & TW_g(y)
\end{array}
\]

**Proposition 3.2.2.** Let \( U \) and \( V \) be open sets in \( \mathbb{R}^k \) and \( \mathbb{R}^l \) respectively. If \( f : U \to V \) is a diffeomorphism then \( k = l \) and the linear map \( df_x \) is invertible (in other words, the Jacobian \( J f_x \) is a non-singular matrix).

**Tangent spaces of manifolds.**

**Example 3.2.3.** To motivate the definition of tangent spaces of manifolds we recall the notion of tangent spaces of surfaces. Consider a parametric surface in \( \mathbb{R}^3 \) given by \( \varphi(u,v) = (x(u,v), y(u,v), z(u,v)) \). Consider a point \( \varphi(u_0, v_0) \) on the surface. Near to \( (u_0, v_0) \) if we fix \( v = v_0 \) and only allow \( u \) to change then we get
a parametric curve \( \varphi(u, v) \) passing through \( \varphi(u_0, v_0) \). The velocity vector of the curve \( \varphi(u, v) \) is a “tangent vector” to the curve at the point \( \varphi(u_0, v_0) \), and is given by the partial derivative with respect to \( u \): \( \frac{\partial \varphi}{\partial u}(u_0, v_0) \). Similarly we have another “tangent vector” \( \frac{\partial \varphi}{\partial v}(u_0, v_0) \). Then the “tangent space” of the surface at \( \varphi(u_0, v_0) \) is the plane spanned the above two tangent vectors (under some further conditions for this notion to be well-defined).

We can think of a manifold as a multi-dimensional surface. Therefore our definition of tangent space of manifold is a natural generalization.

**Definition 3.2.4.** Let \( M \) be an \( m \)-dimensional manifold in \( \mathbb{R}^k \). Let \( x \in M \) and let \( \varphi: U \to M \), where \( U \) is an open set in \( \mathbb{R}^m \), be a parametrization of a neighborhood of \( x \). Assume that \( x = \varphi(u) \) where \( u \in U \). We define the tangent space of \( M \) at \( x \), denoted by \( TM_x \), to be the vector space in \( \mathbb{R}^k \) spanned by the vectors \( \frac{\partial \varphi}{\partial u}(u) \), \( 1 \leq i \leq m \).

Clearly \( TM_x = d\varphi_u(TU_u) = d\varphi_u(\mathbb{R}^m) \).

**Example 3.2.5.** Consider a surface \( z = f(x, y) \). Then the tangent plane at \( (x, y, f(x, y)) \) consists of the linear combinations of the vectors \( (1, 0, f_x(x, y)) \) and \( (0, 1, f_y(x, y)) \).

Consider the circle \( S^1 \). Let \( (x(t), y(t)) \) be any curve on \( S^1 \). The tangent space of \( S^1 \) at \( (x, y) \) is spanned by the velocity vector \( (x'(t), y'(t)) \) if this vector is not 0. Since \( x(t)^2 + y(t)^2 = 1 \), differentiating both sides with respect to \( t \) we get \( x(t)x'(t) + y(t)y'(t) = 0 \), or in other words \( (x'(t), y'(t)) \) is perpendicular to \( (x(t), y(t)) \). Thus the tangent space is perpendicular to the radius.

**Proposition 3.2.6.** The tangent space does not depend on the choice of parametrization.

**Proof.** Consider

\[
\begin{aligned}
\begin{array}{ccc}
\varphi & \xrightarrow{\varphi^{-1} \circ \varphi} & \varphi' \\
U & \xrightarrow{\varphi^{-1} \circ \varphi} & U'
\end{array}
\end{aligned}
\]

where \( \varphi^{-1} \circ \varphi \) is a diffeomorphism. Notice that the map \( \varphi^{-1} \circ \varphi \) is to be understood as follows. We have that \( \varphi(U) \cap \varphi'(U') \) is a neighborhood of \( x \in M \). Restricting to \( \varphi^{-1}(\varphi(U) \cap \varphi'(U')) \), the map \( \varphi^{-1} \circ \varphi \) is well-defined.

The above diagram gives us, with any \( v \in \mathbb{R}^m \):

\[
d\varphi_u(v) = d\varphi'_u(\varphi^{-1} \circ \varphi(u))(d(\varphi^{-1} \circ \varphi)_u(v)).
\]

Thus any tangent vector with respect to the parametrization \( \varphi \) is also a tangent vector with respect to the parametrization \( \varphi' \). We conclude that the tangent space does not depend on the choice of parametrization.

**Proposition 3.2.7 (Tangent space has same dimension as manifold).** If \( M \) is an \( m \)-dimensional manifold then the tangent space \( TM_x \) is an \( m \)-dimensional linear space.
Definition 3.2.8. The derivative of a function \( f \) at \( x \) is defined to be the linear map
\[
d_x f : T_x M \to T_{f(x)} N
\]
\[
h \mapsto d_x f(h) = d_f(x)(h).
\]
Observe that \( d_x f = dF_x|_{TM_x} \).
We need to show that the derivative is well-defined.

Proposition 3.2.9. \( d_x f(h) \in T_{f(x)} N \) and does not depend on the choice of \( F \).

Proof. We have a commutative diagram

\[
\begin{array}{c}
W \xrightarrow{F} N \\
\downarrow_{φ} \quad \downarrow_{ψ} \\
U \xrightarrow{ψ^{-1} φ} V
\end{array}
\]

Assume that \( φ(u) = x, ψ(v) = f(x) \), \( h = dφ_u(w) \). The diagram induces that
\[
d_x f(w) = dF_x(dφ_u(w)) = dF_x(w) = dψ_v(d(ψ^{-1} φ f)(u)(w)).
\]
From this we get the desired conclusion.

Proposition 3.2.10 (The Chain Rule). If \( f : M \to N \) and \( g : N \to P \) are smooth functions between manifolds, then
\[
d(g \circ f)_x = dg_{f(x)} \circ d_f x.
\]

Proof. There is a neighborhood of \( x \) in \( \mathbb{R}^k \) and an extension \( F \) of \( f \) to that neighborhood. Similarly there is a neighborhood \( y \) in \( \mathbb{R}^l \) and an extension \( G \) of \( g \) to that neighborhood. Then
\[
d(g \circ f)_x = d(G \circ F)_x|_{TM_x} = (dG_y \circ dF_x)|_{TM_x} = dG_y|_{TN_y} \circ dF_x|_{TM_x} = dg_y \circ dF_x.
\]

Definition 3.2.11. If \( M \) and \( N \) are two smooth manifolds in \( \mathbb{R}^k \) and \( M \subset N \) then we say that \( M \) is a submanifold of \( N \).

Proposition 3.2.12. If \( f : M \to N \) is a diffeomorphism then \( d_x f : TM_x \to T_{f(x)} N \) is a linear isomorphism. In particular the dimensions of the two manifolds are same.

Proof. Let \( m = \dim M \) and \( n = \dim N \). Since \( d_x f \circ d_{f(x)}^{-1} = Id_{TN_x} \) and \( d_{f(x)}^{-1} \circ d_x f = Id_{TM_x} \) we deduce, via the rank of \( d_x f \) that \( m \geq n \). Doing the same with \( d_{f(x)}^{-1} \) we get \( m \leq n \), hence \( m = n \). From that \( d_x f \) must be a linear isomorphism.

Corollary 3.2.13. \( \mathbb{R}^k \) and \( \mathbb{R}^l \) are not diffeomorphic if \( k \neq l \).
Problems.

3.2.14. Calculate the tangent spaces of $S^n$.

3.2.15. Calculate the tangent spaces of the hyperboloid $x^2 + y^2 - z^2 = a$, $a > 0$.

3.2.16. If $\text{Id} : M \to M$ is the identify map then $d(\text{Id})_x$ is $\text{Id} : T_Mx \to T_Mx$.

3.2.17. If $M$ is a submanifold of $N$ then $T_Mx$ is a subspace of $T_Nx$.

3.2.18. * In general, a curve on a manifold $M$ is a smooth map $c$ from an open interval of $\mathbb{R}$ to $M$. The derivative of this curve is a linear map $\frac{dc}{dt}(t_0) : \mathbb{R} \to TM_{c(t_0)}$, represented by the vector $c'(t_0) \in \mathbb{R}^k$, this vector is called the velocity vector of the curve at $t = t_0$.

Show that any vector in $T_Mx$ is the velocity vector of a curve in $M$.

3.2.19. Show that if $M$ and $N$ are manifolds and $M \subset N$ then $T_Mx \subset T_Nx$.

3.2.20. (a) Calculate the derivative of the map $f : (0, 2\pi) \to S^1$, $f(t) = (\cos t, \sin t)$.

(b) Calculate the derivative of the map $f : S^1 \to \mathbb{R}, f(x, y) = e^y$.

3.2.21 (Cartesian products of manifolds). * If $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ are manifolds then $X \times Y \subset \mathbb{R}^{k+l}$ is also a manifold. Furthermore $T(X \times Y)_{(x,y)} = TX_x \times TY_y$. 
3.3. Regular Values

Let $M$ be a manifold of dimension $m$ and $N$ be manifold of dimension $n$, and let $f : M \to N$ be smooth.

A point $x \in M$ is called a **regular point** of $f$ if $df_x$ is surjective. If $df_x$ is not surjective then we say that $x$ is a **critical point** of $f$.

If $x$ is a critical point of $f$ then its value $y = f(x)$ is called a **critical value** of $f$.

A value $y \in N$ is called a **regular value** if it is not a critical value, that is if $f^{-1}(\{y\})$ contains only regular points.

For brevity we will write $f^{-1}(\{y\})$ as $f^{-1}(y)$.

Thus $y$ is a critical value if and only if $f^{-1}(y)$ contains a critical point. In particular, if $y \in N \setminus f(M)$ then $f^{-1}(y) = \emptyset$ but $y$ is still considered a regular value.

**Example 3.3.1.** The function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$ has $(0, 0)$ as the only critical point.

**Example 3.3.2.** If $f : M \to N$ where $\dim(M) < \dim(N)$ then every $x \in M$ is a critical point and every $y \in N$ is a critical value.

**Example 3.3.3.** Let $U$ be an open interval in $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be smooth. Then $x \in U$ is a critical point of $f$ if and only if $f'(x) = 0$.

**Example 3.3.4.** Let $U$ be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be smooth. Then $x \in U$ is a critical point of $f$ if and only if $\nabla f(x) = 0$.

**The Inverse Function Theorem and the Implicit Function Theorem.** First we state the Inverse Function Theorem in Multivariables Calculus.

**Theorem 3.3.5 (Inverse Function Theorem).** Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be smooth. If $df_x$ is bijective then $f$ is locally a diffeomorphism.

More concisely, if $\det(J f_x) \neq 0$ then there is an open neighborhood $U$ of $x$ and an open neighborhood $V$ of $f(x)$ such that $f|_U : U \to V$ is a diffeomorphism.

**Remark 3.3.6.** For a proof, see for instance [Spi65]. Usually the result is stated for continuously differentiable function (i.e. $C^1$), but the result for smooth functions follows from that one, since the Jacobian matrix of the inverse map is the inverse matrix of the Jacobian of the original map, and the entries of an inverse matrix can be obtained from the entries of the original matrix via smooth operations, namely

\[
A^{-1} = \frac{1}{\det A} A^*,
\]

here $A^*_{i,j} = (-1)^{i+j} \det(A^i_j)$, and $A^i_j$ is obtained from $A$ by omitting the $i$th row and $j$th column.

**Theorem 3.3.7 (Implicit Function Theorem).** Suppose that $f : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is smooth and $f(x) = y$. If $df_x$ is onto then locally at $x$ the level set $f^{-1}(y)$ is a graph of dimension $m$. 
More concisely, suppose that $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth and $f(x_0, y_0) = 0$. If the matrix $[D^m f_i(x_0, y_0)]$, $1 \leq i, j \leq n$ is non-singular then there is a neighborhood $U \times V$ of $(x_0, y_0)$ such that for each $x \in U$ there is a unique $g(x) \in V$ satisfying $f(x, g(x)) = 0$. The function $g$ is smooth.

The Implicit Function Theorem is obtained by setting $F(x, y) = (x, f(x, y))$ and applying the Inverse Function Theorem to $F$.

**Theorem 3.3.8 (Inverse Function Theorem for manifolds).** Let $M$ and $N$ be two manifolds of the same dimensions, and let $f : M \to N$ be smooth. If $x$ is a regular point of $f$ then there is a neighborhood in $M$ of $x$ on which $f$ is a diffeomorphism onto its image.

**Proof.** Consider

\[
\begin{array}{c}
M \\ \downarrow \varphi \\
U \\
\quad \psi^{-1} \circ f \circ \varphi \\
\downarrow \\
V \\
\end{array}
\quad
\begin{array}{c}
N \\ \downarrow \psi \\
W \\
\end{array}
\]

Since $d f_x$ is surjective, it is bijective. Then $d(\psi^{-1} \circ f \circ \varphi)_u = d\psi^{-1}_f \circ d f_x \circ d \varphi_u$ is an isomorphism. The Inverse Function Theorem can be applied to $\psi^{-1} \circ f \circ \varphi$, giving that it is a local diffeomorphism at $u$, so $f$ is a local diffeomorphism at $x$. □

**Preimage of a regular value.**

**Corollary 3.3.9.** If $\dim(M) = \dim(N)$ and $y$ is a regular value of $f$ then $f^{-1}(y)$ is a discrete set. In other words, $f^{-1}(y)$ is a zero dimensional manifold. Furthermore if $M$ is compact then $f^{-1}(y)$ is a finite set.

**Proof.** If $x \in f^{-1}(y)$ then there is a neighborhood of $x$ on which $f$ is a bijection. That neighborhood contains no other point in $f^{-1}(y)$. Thus $f^{-1}(y)$ is a discrete set.

If $M$ is compact then the set $f^{-1}(y)$ is compact. If it the set is infinite then it has a limit point $x_0$. Because of the continuity of $f$, we have $f(x_0) = y$. That contradicts the fact that $f^{-1}(y)$ is discrete. □

**Theorem 3.3.10 (Preimage of a regular value is a manifold).** If $y$ is a regular value of $f : M \to N$ then $f^{-1}(y)$ is a manifold of dimension $\dim(M) - \dim(N)$.

The theorem is essentially the Implicit Function Theorem carried over to manifolds.

**Proof.** Let $m = \dim(M)$ and $n = \dim(N)$. The case $m = n$ is already considered in 3.3.9. Now we assume $m > n$. Let $x_0 \in f^{-1}(y_0)$. Consider the diagram

\[
\begin{array}{c}
M \\ \downarrow \varphi \\
O \\
\quad \psi \\
\downarrow \\
N \\
\end{array}
\quad
\begin{array}{c}
W \\
\end{array}
\]
where \( g = \psi^{-1} \circ f \circ \varphi \) and \( \psi(w_0) = y_0 \).

Since \( df_{x_0} \) is onto, \( d\psi^{-1}(x_0) \) is also onto. If needed we can change \( g, O \) and \( \varphi \) by permuting variables appropriately such that the matrix \( [Dg_i(\psi^{-1}(x_0))] \), \( 1 \leq i \leq n \), \( m - n + 1 \leq j \leq m \) is non-singular. Denote \( \psi^{-1}(x_0) = (u_0, v_0) \in \mathbb{R}^{m-n} \times \mathbb{R}^n \).

By the Implicit Function Theorem applied to \( g \) there is an open neighborhood \( U \) of \( u_0 \) in \( \mathbb{R}^{m-n} \) and an open neighborhood \( V \) of \( v_0 \) in \( \mathbb{R}^n \) such that \( U \times V \) is contained in \( O \) and on \( U \times V \) we have \( g(u, v) = w_0 \) if and only if \( v = h(u) \) for a certain smooth function \( h : U \to V \). In other words, on \( U \times V \) the equation \( g(u, v) = w_0 \) determines a graph \( (u, h(u)) \).

Now we have \( \varphi(U \times V) \cap f^{-1}(y_0) = \{ \varphi(u, h(u)) \mid u \in U \} \). Let \( \tilde{\varphi}(u) = \varphi(u, h(u)) \) then \( \tilde{\varphi} \) is a diffeomorphism from \( U \) onto \( \varphi(U \times V) \cap f^{-1}(y_0) \), a neighborhood of \( x_0 \) in \( f^{-1}(y_0) \). \( \Box \)

**Example 3.3.11.** To be able to follow the proof more easily the reader can try to work it out for an example, such as the case where \( M \) is the graph of the function \( z = x^2 + y^2 \), and \( f \) is the height function \( f((x, y, z)) = z \) defined on \( M \).

The \( n \)-sphere \( S^n \) is a subset of \( \mathbb{R}^{n+1} \) determined by the implicit equation \( \sum_{i=1}^{n+1} x_i^2 = 1 \). Since 1 is a regular value of the function \( f(x_1, x_2, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2 \) we conclude that \( S^n \) is a manifold of dimension \( n \).

**Lie groups.** The set \( M_n(\mathbb{R}) \) of \( n \times n \) matrices over \( \mathbb{R} \) can be identified with the Euclidean manifold \( \mathbb{R}^{n^2} \).

Consider the map \( \det : M_n(\mathbb{R}) \to \mathbb{R} \). Let \( A = [a_{ij}] \in M_n(\mathbb{R}) \). Since \( \det(A) = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} = \sum_j (-1)^{i+j} a_{ij} \det(A^{i,j}) \), we can see that \( \det \) is a smooth function.

Let us find the critical points of \( \det \). A critical point is a matrix \( A = [a_{ij}] \) at which \( \frac{\partial \det(A)}{\partial a_{ij}}(A) = (-1)^{i+j} \det(A^{i,j}) = 0 \) for all \( i, j \). In particular, \( \det(A) = 0 \). So 0 is the only critical value of \( \det \).

Therefore \( SL_n(\mathbb{R}) = \det^{-1}(1) \) is a manifold of dimension \( n^2 - 1 \).

Furthermore we note that the group multiplication in \( SL_n(\mathbb{R}) \) is a smooth map from \( SL_n(\mathbb{R}) \times SL_n(\mathbb{R}) \) to \( SL_n(\mathbb{R}) \). The inverse operation is a smooth map from \( SL_n(\mathbb{R}) \) to itself, because of for example Formula (3.3.1). We then say that \( SL_n(\mathbb{R}) \) is a Lie group.

**Definition 3.3.12.** A *Lie group* is a smooth manifold which is also a group, for which the group operations are compatible with the smooth structure, namely the group multiplication and inversion are smooth.

Let \( O(n) \) be the group of orthogonal \( n \times n \) matrices, the group of linear transformation of \( \mathbb{R}^n \) that preserves distances.

**Proposition 3.3.13.** The orthogonal group \( O(n) \) is a manifold. Further, it is a Lie group.

**Proof.** Let \( S(n) \) be the set of symmetric \( n \times n \) matrices. This is clearly a manifold of dimension \( \frac{n^2 + n}{2} \).
Consider the smooth map \( f : M(n) \to S(n), f(A) = AA^t \). We have \( O(n) = f^{-1}(I) \). We will show that \( I \) is a regular value of \( f \).

We compute the derivative of \( f \) at \( A \in f^{-1}(I) \):

\[
df_A(B) = \lim_{t \to 0} \frac{f(A + tB) - f(A)}{t} = BA^t + AB^t.
\]

We note that the tangents spaces of \( M(n) \) and \( S(n) \) are themselves. To check whether \( df_A \) is onto for \( A \in O(n) \), we need to check that given \( C \in S(n) \) there is a \( B \in M(n) \) such that \( C = BA^t + AB^t \). We can write \( C = \frac{1}{2}C + \frac{1}{2}C \), and the equation \( \frac{1}{2}C = BA^t \) will give a solution \( B = \frac{1}{2}CA \), which is indeed a solution to the original equation. \( \square \)

**Problems.**

3.3.14. Let \( f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = x^2 - y^2 \). Show that if \( a \neq 0 \) then \( f^{-1}(a) \) is a 1-dimensional manifold, but \( f^{-1}(0) \) is not.

3.3.15. Show that if \( a \) and \( b \) are both positive or both negative then \( f^{-1}(a) \) and \( f^{-1}(b) \) are diffeomorphic.

3.3.16. Let \( f : \mathbb{R}^3 \to \mathbb{R}, f(x, y) = x^2 + y^2 - z^2 \). Show that if \( a \neq 0 \) then \( f^{-1}(a) \) is a 2-dimensional manifold, but \( f^{-1}(0) \) is not.

Show that if \( a \) and \( b \) are both positive or both negative then \( f^{-1}(a) \) and \( f^{-1}(b) \) are diffeomorphic.

3.3.17. Show that the height function on the sphere \( S^2 \) has exactly two critical points. The height function is the function \( (x, y, z) \mapsto z \).

3.3.18. If \( x \) is a local extremum of \( f \) then \( x \) is a critical point of \( f \).

3.3.19. If \( M \) is a compact manifold, \( \dim(M) \geq 1 \), and \( f \) is a smooth real function on \( M \) then \( f \) has at least two critical points.

3.3.20. Let \( \dim(M) = \dim(N) \), \( M \) be compact and \( S \) be the set of all regular values of \( f : M \to N \). For \( y \in S \), let \( |f^{-1}(y)| \) be the number of elements of \( f^{-1}(y) \). The map

\[
\begin{align*}
S & \to N \\
y & \mapsto |f^{-1}(y)|.
\end{align*}
\]

is locally constant.

**Hint:** Each \( x \in f^{-1}(y) \) has a neighborhood \( U_x \) on which \( f \) is a diffeomorphism. Let \( V = [\bigcap_{x \in f^{-1}(y)} f(U_x)] \setminus f(M \setminus \bigcup_{x \in f^{-1}(y)} U_x) \). Consider \( V \cap S \).

3.3.21. Use regular value to show that the torus \( T^2 \) is a manifold.

Find the regular values of the function \( f(x, y, z) = [4x^2(1 - x^2) - y^2]^2 + z^2 - \frac{1}{4} \).

3.3.22. Find a counter-example to show that 3.3.9 is not correct if regular value is replaced by critical value.

3.3.23. * If \( f : M \to N \) is smooth and \( x \in f^{-1}(y) \) is a regular point of \( f \) then the kernel of \( df_x \) is exactly \( T f^{-1}(y) \).

3.3.24. Show that \( S^1 \) is a Lie group.
3.3.25. Show that the set of all invertible \( n \times n \)-matrices \( GL(n; \mathbb{R}) \) is a Lie group and find its dimension.

3.3.26. (a) Check that the derivative of the determinant map \( \det : M_n(\mathbb{R}) \to \mathbb{R} \) is represented by a gradient vector whose \((i, j)\)-entry is \((-1)^{i+j} \det(A_{i,j})\).

(b) Determine the tangent space of \( SL_n(\mathbb{R}) \) at \( A \in SL_n(\mathbb{R}) \).

*Hint: Use Problem 3.3.23.)*

(c) Let \( I \) be the \( n \times n \) identity matrix. Show that the tangent space of \( SL_n(\mathbb{R}) \) at the identity element \( I \) is the set of all \( n \times n \) matrices with zero traces. This is called the Lie algebra \( sl_n(\mathbb{R}) \) of the Lie group \( SL_n(\mathbb{R}) \).

In general if \( G \) is a Lie group then the tangent space of \( G \) at its identity element is called the *Lie algebra* of \( G \), often denoted by \( g \).
3.4. Manifolds with Boundary

Example 3.4.1. The closed disk $D^n$ is an $n$-manifold with boundary.

Consider the closed half-space $H^m = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\}$. The boundary of $H^m$ is $\partial H^m = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_m = 0\} = \mathbb{R}^{m-1} \times \{0\}$.

Definition 3.4.2. A subset $M$ of $\mathbb{R}^k$ is called a manifold with boundary of dimension $m$ if each point in $M$ has a neighborhood which is diffeomorphic to an open neighborhood of $H^m$.

The boundary $\partial M$ of $M$ is the set of all points of $M$ which are mapped to $\partial H^m$ under the above diffeomorphisms. The complement of the boundary is called the interior.

We need to check that our definition of the boundary does not depend on the choice of the parametrizations. An inspection shows that the question is reduced to:

Lemma 3.4.3. Suppose that $f$ is a diffeomorphism from the closed unit disk $D^n = B'(0, 1)$ in $\mathbb{R}^n$ to itself. Then $f$ maps interior points to interior points.

Proof. Suppose that $x$ is an interior point of $D^n$. Consider $f|_{B(0,1)} : \mathbb{R}^n \to \mathbb{R}^n$, a function defined on an open set in $\mathbb{R}^n$. By the chain rule $df_x$ is non-singular, therefore by the Inverse Function Theorem $f$ is a diffeomorphism from an open ball containing $x$ onto an open ball containing $f(x)$. Thus $f(x)$ must be an interior point.

Remark 3.4.4. Alternatively we can use Invariance of dimension 1.5.51.

Remark 3.4.5. The boundary of a manifold is not its topological boundary.

From now on when we talk about a manifold it may be with or without boundary.

Proposition 3.4.6. Let $M$ be an $m$-manifold with boundary. The boundary of $M$ is an $(m - 1)$-manifold without boundary, and the interior of $M$ is an $m$-manifold without boundary.

Proof. Let $x \in \partial M$. There is a parametrization $\varphi$ from a neighborhood $U$ of $0$ in $\partial H^m$ to a neighborhood $V$ of $x$ in $M$, such that $\varphi(0) = x$. By definition, $\varphi(U \cap \partial H^m) = V \cap \partial M$. Therefore $\varphi|_{U \cap \partial H^m}$ is a parametrization of a neighborhood of $x$ in $\partial M$. Thus $\partial M$ is an $(m - 1)$-manifold without boundary.

Example 3.4.7. Let $f$ be the height function on $S^2$ and let $y$ be a regular value. Then the set $f^{-1}((-\infty, y])$ is a disk with the circle $f^{-1}(y)$ as the boundary.

Theorem 3.4.8. Let $M$ be an $m$-dimensional manifold without boundary. Let $f : M \to \mathbb{R}$ be smooth. Let $y$ be a regular value of $f$. Then $f^{-1}(y, \infty)$ is an $m$-dimensional manifold with boundary $f^{-1}(y)$. 

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PROOF. Let $N = f^{-1}(y, \infty)$. If $x \in M$ such that $f(x) > y$ then there is a neighborhood $O$ of $x$ in $M$ such that $f(O) \subset [y, \infty)$. Since $O \subset N$, we conclude that $x$ is an interior point of $N$.

The crucial case is when $f(x) = y$. As in the proof of 3.3.10, there is an open ball $U$ in $\mathbb{R}^m$ containing $0$ and an open interval $V$ in $\mathbb{R}$ containing $0$ such that in $U \times V$ the set $g^{-1}(y)$ is a graph $\{(u, v) \mid v = h(u), u \in U\}$. Then a neighborhood in $M$ of $x$ is parametrized as $\varphi(u, v)$ with $(u, v) \in U \times V$, and in that neighborhood $f^{-1}(y)$ is parametrized as $\varphi(u, h(u))$ with $u \in U$.

![Figure 3.4.1.](image)

Since $(U \times V) \setminus g^{-1}(y)$ consists of two connected components, exactly one of the two is mapped via $g$ to $(y, \infty)$. In order to be definitive, let us assume that it is $W = \{(u, v) \mid v \geq h(u)\}$ that is mapped to $[y, \infty)$. Then $\varphi : W \to N$ is a parametrization of a neighborhood of $x$ in $N$. It is not difficult to see that $W$ is diffeomorphic to an open neighborhood of $0$ in $\mathbb{H}^m$. In fact the map $W \to \mathbb{H}^m$ given by $(u, v) \mapsto (u, v - h(u))$ would give the desired diffeomorphism. Thus $x$ is a boundary point of $N$. □

The tangent space of a manifold with boundary $M$ is defined as follows. If $x$ is an interior point of $M$ then $TM_x$ is defined as before. If $x$ is a boundary point then there is a parametrization $\varphi : U \to M$, where $U$ is an open neighborhood of $0$ in $\mathbb{H}^m$ and $\varphi(0) = x$. Then $TM_x$ is still defined as $d\varphi_0(\mathbb{R}^m)$.

**Example 3.4.9.** If $y$ is a regular value of the height function on $D^2$ then $f^{-1}(y)$ is a 1-dimensional manifold with boundary on $\partial D^2$. 
**Theorem 3.4.10.** Let $M$ be an $m$-dimensional manifold with boundary, let $N$ be an $n$-dimensional manifold with or without boundary. Let $f : M \to N$ be smooth. Suppose that $y \in N$ is a regular value of both $f$ and $f|_{\partial M}$. Then $f^{-1}(y)$ is an $(m - n)$-manifold with boundary $\partial M \cap f^{-1}(y)$.

**Proof.** That $f^{-1}(y) \setminus \partial M$ is an $(m - n)$-manifold without boundary is already proved in 3.3.10.

We consider the crucial case of $x \in \partial M \cap f^{-1}(y)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.4.2.png}
\caption{Figure 3.4.2.}
\end{figure}

The map $g$ can be extended to $\tilde{g}$ defined on an open neighborhood $\tilde{U}$ of 0 in $\mathbb{R}^m$. As before, $\tilde{g}^{-1}(y)$ is a graph of a function of $(m - n)$ variables so it is an $(m - n)$-manifold without boundary.

Let $p : \tilde{g}^{-1}(y) \to \mathbb{R}$ be the projection to the last coordinate (the height function). We have $g^{-1}(y) = p^{-1}([0, \infty))$ therefore if we can show that 0 is a regular value of $p$ then the desired result follows from 3.4.8 applied to $\tilde{g}^{-1}(y)$ and $p$. For that we need to show that the tangent space $T_{\tilde{g}^{-1}(y)}u$ at $u \in p^{-1}(0)$ is not contained in $\partial \mathbb{H}^m$. Note that since $u \in p^{-1}(0)$ we have $u \in \partial \mathbb{H}^m$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.4.3.png}
\caption{Figure 3.4.3.}
\end{figure}
Since $\tilde{g}$ is regular at $u$, the null space of $d\tilde{g}_u$ on $T\tilde{U}_u = \mathbb{R}^m$ is exactly $T\tilde{g}^{-1}(y)_u$, of dimension $m - n$. On the other hand, $\tilde{g}|_{\partial H^m}$ is regular at $u$, which implies that the null space of $d\tilde{g}_u$ restricted to $T(\partial H^m)_u = \partial H^m$ has dimension $(m - 1) - n$. Thus $T\tilde{g}^{-1}(y)_u$ is not contained in $\partial H^m$. □

Problems.

3.4.11. Show that the subspace $\{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_m > 0\}$ is diffeomorphic to the space to $\mathbb{R}^m$. 
3.5. The Brouwer Fixed Point Theorem

We use the following theorem of A. Sard, which belongs to Analysis:

**Theorem 3.5.1. Sard Theorem** Let $U$ be an open subset of $\mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ be smooth. Then the set of critical values of $f$ is of Lebesgue measure zero.

See for instance [Mil97] for a proof.

On manifolds, we have:

**Theorem 3.5.2.** If $M$ and $N$ are two manifolds and $f : M \to N$ is smooth then $f$ has a regular value in $N$. In fact the set of all regular values of $f$ is dense in $N$.

**Proof.** Consider a neighborhood $V$ in $N$ parametrized by a diffeomorphism $\psi : V' \to V$. Consider any neighborhood $U$ in $M$ parametrized by a diffeomorphism $\varphi : U' \to U$ such that $f(U) \subset V$. We have a commutative diagram:

$$
\begin{array}{ccc}
U & \xrightarrow{f} & V \\
\varphi \downarrow & & \downarrow \psi \\
U' & \xrightarrow{g} & V'
\end{array}
$$

From this diagram, if $x$ is a critical point of $f$ then $u = \varphi^{-1}(x)$ is a critical point of $g$, and $g(u) = v$ is a critical value of $g$, and $\psi(v) = f(x)$ is a critical value of $f$.

Let $C$ be the set of all critical points of $f$. Let $A_U$ be the set of all critical values of $g$. By Sard Theorem, $A_U$ is a set of measure zero in $\mathbb{R}^n$.

The set of all critical values of $f$ which are values of $f$ on $U$ is exactly $f(C \cap U) = \psi(A_U)$.

The set of all critical values of $f$ which lie in $V$ is the union of the sets $f(C \cap U)$ over all possible neighborhoods $U$.

We can a cover $M$ by a countable collection $I$ of parametrized neighborhoods. This is possible because a Euclidean space has a countable topological basis (see [1.2.27]), and so does a subset. From $I$ we form the collection $J = \{f^{-1}(V) \cap W \mid W \in I\}$. For each $U \in J$ we have $f(U) \subset V$.

Now we have $f(C) \cap V = \bigcup_{U \in J} f(C \cap U) = \bigcup_{U \in J} \psi(A_U) = \psi(\bigcup_{U \in J} A_U)$.

Since a countable union of sets of measure zero is a set of measure zero, we have $\bigcup_{U \in J} A_U$ is of measure zero. Therefore this set cannot have non-empty interior. Since $\psi$ is a homeomorphism, $\psi(\bigcup_{U \in J} A_U)$ cannot have non-empty interior either. Thus $f(C) \cap V \subseteq V$, so there is a regular value of $f$ in $V$. \qed

If $N \subset M$ and $f : M \to N$ such that $f|_N = \text{id}_N$ then $f$ is called a **retraction** from $M$ to $N$ and $N$ is a **retract** of $M$.

**Lemma 3.5.3.** Let $M$ be a compact manifold with boundary. There is no smooth map $f : M \to \partial M$ such that $f|_{\partial M} = \text{id}_{\partial M}$. In other words there is no smooth retraction from $M$ to its boundary.
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PROOF. Suppose that there is such a map \( f \). Let \( y \) be a regular value of \( f \). Since \( f|_{\partial M} \) is the identity map, \( y \) is also a regular value of \( f|_{\partial M} \). By Theorem 3.4.10 the inverse image \( f^{-1}(y) \) is a 1-manifold with boundary \( f^{-1}(y) \cap \partial M = \{y\} \). But a 1-manifold cannot have boundary consisting of exactly one point. This result is contained in the classification of compact one-dimensional manifolds. □

**Theorem 3.5.4 (Classification of compact one-dimensional manifolds).** A smooth compact connected one-dimensional manifold is diffeomorphic to either a circle, in which case it has no boundary, or an arc, in which case its boundary consists of two points.

See [Mil97] for a proof.

**Theorem 3.5.5 (Smooth Brouwer Fixed Point Theorem).** A smooth map from the disk \( D^n \) to itself has a fixed point.

PROOF. Suppose that \( f \) does not have a fixed point, i.e. \( f(x) \neq x \) for all \( x \in D^n \). The line from \( f(x) \) to \( x \) will intersect the boundary \( \partial D^n \) at a point \( g(x) \). Then \( g : D^n \to \partial D^n \) is a smooth function and is the identity on \( \partial D^n \). That is impossible, by 3.5.3 □

Actually the Brouwer Fixed Point Theorem holds true for continuous map. The proof use the smooth version of the theorem. Since its proof does not use smooth techniques we will not present it here, the interested reader can find it in Milnor’s book [Mil97]. Basically one approximates a continuous function by smooth ones.

**Theorem 3.5.6 (Brouwer Fixed Point Theorem).** A continuous map from the disk \( D^n \) to itself has a fixed point.

**Problems.**

3.5.7. Show that a smooth loop on \( S^2 \) (i.e. a smooth map from \( S^1 \) to \( S^2 \)) cannot cover \( S^2 \). Also, there is no smooth surjective map from \( \mathbb{R} \) to \( \mathbb{R}^n \) with \( n > 1 \). In other words, there is no smooth space filling curves, in contrast to the continuous case (compare 1.5.54).

3.5.8. Prove the Brouwer fixed point theorem for \([0, 1] \) directly.

3.5.9. Check that the function \( g \) in the proof of 3.5.5 above is smooth.

3.5.10. Show that the Brouwer fixed point theorem is not correct for \((0, 1) \).

3.5.11. Show that the Brouwer fixed point theorem is not correct for open balls in \( \mathbb{R}^n \).

**Hint:** Let \( \varphi \) be a diffeomorphism from \( B^n \) to \( \mathbb{R}^n \). Let \( f \) be any smooth map on \( \mathbb{R}^n \). Consider the map \( \varphi^{-1} f \varphi \).

3.5.12. Find a smooth map from the solid torus to itself without fixed point.

3.5.13. Suppose that \( M \) is diffeomorphic (or homeomorphic) to \( D^n \). Show that a smooth (or continuous) map from \( M \) to itself has a fixed point.

3.5.14. Let \( A \) be an \( n \times n \) matrix whose entries are all nonnegative.

(1) Check that the map \( \tilde{A} : S^{n-1} \to S^{n-1}, v \mapsto \frac{Av}{||Av||} \) brings the first quadrant \( Q = \{(x_1, x_2, \ldots, x_n) \in S^{n-1} \mid x_i \geq 0, 1 \leq i \leq n\} \) to itself.
(2) Prove that $Q$ is homeomorphic to the closed ball $D^{n-1}$.

(3) Derive a theorem of Frobenius that $A$ must have a nonnegative eigenvalue.
3.6. Orientation

**Orientation on vector spaces.** On a finite dimensional real vector space, two bases are said to determine the same orientation of the space if the change of bases matrix has positive determinant. Being of the same orientation is an equivalence relation on the set of all bases. With this equivalence relation the set of all bases is divided into two equivalence classes. If we choose one of the two as the positive one, then we say the vector space is *oriented* and the chosen equivalence class of bases is called the *orientation* (or positive orientation). Thus any finite dimensional real vector space is *orientable* (i.e. can be oriented) with two possible orientations.

**Example 3.6.1.** The standard positive orientation of $\mathbb{R}^n$ is represented by the basis

$$\{e_1 = (1,0,\ldots,0), e_2 = (0,1,0,\ldots,0), \ldots, e_n = (0,\ldots,0,1)\}.$$  

Let $T$ be an isomorphism from an oriented finite dimensional real vector space $V$ to an oriented finite dimensional real vector space $W$. Then $T$ brings a basis of $V$ to a basis of $W$.

There are only two possibilities. Either $T$ brings a positive basis of $V$ to a positive basis of $W$, or $T$ brings a positive basis of $V$ to a negative basis of $W$. In the first case we say that $T$ is *orientation-preserving*, and in the second case we say that $T$ is *orientation-reversing*.

**Orientation on manifolds.** A manifold $M$ is said to be oriented if for each point $x \in M$ an orientation for $T_xM$ is chosen and each point has a neighborhood that can be parametrized by a diffeomorphism $\varphi: U \to M$ such that the derivative $d\varphi_u : \mathbb{R}^m \to T_xM$ brings the standard positive orientation of $\mathbb{R}^m$ to the chosen orientation of $T_xM$.

Equivalently, a manifold is said to be oriented if there is a consistent way to orient the tangent spaces of the manifold using parametrizations. Each parametrization induces an orientation of the tangent space. Consistency (i.e. being well-defined) means that when two parametrized neighborhood overlap the orientations on the tangent spaces induced by the parametrizations must be the same.

Concisely, suppose that $\varphi : U \to M$ is a parametrization of a neighborhood in $M$. At each point, the orientation on $T_xM$ is given by the image of the standard basis of $\mathbb{R}^n$ via $d\varphi_u$, i.e. it is given by the basis $\{\frac{\partial\varphi}{\partial u_1}(e_1), \frac{\partial\varphi}{\partial u_2}(u) = d\varphi_u(e_2), \ldots, \frac{\partial\varphi}{\partial u_n}(u) = d\varphi_u(e_n)\}$. Suppose that $\psi : V \to M$ parametrizes an overlapping neighborhood. Then $\psi$ gives an orientation of $T_xM$ by the basis $\{\frac{\partial\psi}{\partial v_1}(v) = d\psi_v(e_1), \frac{\partial\psi}{\partial v_2}(v) = d\psi_v(e_2), \ldots, \frac{\partial\psi}{\partial v_n}(v) = d\psi_v(e_n)\}$. Since $d\psi_v = d(\psi \circ \varphi^{-1})_v \circ d\varphi_u$, the consistency requirement is that the determinant of the change of base matrix $d(\varphi' \circ \varphi^{-1})_v$ must be positive. We can say that the change of coordinates must be orientation preserving.

If a manifold could be oriented then we say that it is *orientable*. A connected orientable manifold could be oriented in two ways.
Orientable Surfaces. A surface is two-sided if there is a smooth way to choose a unit normal vector \( N(p) \) at each point \( p \in S \). That is, the map \( N : S \to \mathbb{R}^3 \) is smooth.

![Mobius band](image.png)

**Figure 3.6.1.** The Mobius band is not orientable and is one-sided.

**Proposition 3.6.2.** A surface is orientable if and only if it is two-sided.

**Proof.** If the surface is orientable then its tangent spaces could be oriented smoothly. That means the unit normal vector \( \frac{r_u \times r_v}{|r_u \times r_v|} \) is well-defined on the surface, and it is certainly smooth.

Conversely, if there is a smooth unit normal vector \( N(p) \) on the surface then we can parametrize a neighborhood of any point by a parametrization \( r(u, v) \) such that \( r_u \times r_v \) is in the same direction with \( N \) (if the given parametrization gives the reversed orientation we can just switch the variables \( u \) and \( v \)). This can be done if we choose the neighborhood to be connected, since then \( \langle r_u \times r_v, N \rangle \) will always be positive in that neighborhood. \( \square \)

**Example 3.6.3.** The sphere is orientable.

**Example 3.6.4.** A graph is an orientable surface. Any surface which could be parametrized by one parametrization is orientable.

**Proposition 3.6.5.** If \( f : \mathbb{R}^3 \to \mathbb{R} \) is smooth and \( a \) is a regular value of \( f \) then \( f^{-1}(a) \) is an orientable surface.

**Proof.** In Multi-variable Calculus we know that the gradient vector \( \nabla f(p) \) is always perpendicular to the level set containing \( p \). \( \square \)

**Example 3.6.6.** The torus is orientable. This can be seen from an implicit equation of the torus.

**Orientation on the boundary of an oriented manifold.** Suppose that \( M \) is a manifold with boundary and the interior of \( M \) is oriented. We orient the boundary of \( M \) as follows. Suppose that under an orientation-preserving parametrization \( \varphi \) the point \( \varphi(x) \) is on the boundary \( \partial M \) of \( M \). Then the orientation \( \{b_2, b_3, \ldots, b_n\} \) of \( \partial \mathbb{H}^n \) such that the ordered set \( \{-e_n, b_2, b_3, \ldots, b_n\} \) is a positive basis of \( \mathbb{R}^n \) will
induce the positive orientation for $T\partial M_{\varphi(x)}$ through $d\varphi(x)$. This is called the outer normal first orientation of the boundary.

Problems.

3.6.7. Two diffeomorphic surfaces are either both orientable or both un-orientable.

*Hint:* $d(f \circ r) = df \circ dr$ and $r^{-1} \circ (f \circ r) = (f \circ r')^{-1} \circ (f \circ r)$. 
3.7. Brouwer Degree

Let $M$ and $N$ be boundaryless, oriented manifolds of the same dimensions $m$. Further suppose that $M$ is compact.

Let $f : M \to N$ be smooth. Suppose that $x$ is a regular point of $f$. Then $df_x$ is an isomorphism from $TM_x$ to $TN_{f(x)}$. Let $\text{sign}(df_x) = 1$ if $df_x$ preserves orientations, and $\text{sign}(df_x) = -1$ otherwise.

For any regular value $y$ of $f$, let

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \text{sign}(df_x).$$

Notice that the set $f^{-1}(y)$ is a finite because $M$ is compact (see 3.3.9).

This number $\deg(f, y)$ is called the Brouwer degree (bậc Brouwer) or topological degree of the map $f$ with respect to the regular value $y$.

From the Inverse Function Theorem 3.3.8, each regular value $y$ has a neighborhood $V$ and each preimage $x$ of $y$ has a neighborhood $U_x$ on which $f$ is a diffeomorphism onto $V$, either preserving or reversing orientation. Therefore we can interpret that $\deg(f, y)$ counts the algebraic number of times the function $f$ covers the value $y$.

Example 3.7.1. Consider $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$. Then $\deg(f, 1) = 0$. This could be explained geometrically from the graph of $f$, as $f$ covers the value 1 twice in opposite directions at $x = -1$ and $x = 1$.

On the other hand, consider $g(x) = x^3 - x$ with the regular value 0. From the graph of $f$ we see that $f$ covers the value 0 three times with positive signs at $x = -1$ and $x = 1$ and negative sign at $x = 0$, therefore we see rightaway that $\deg(g, 0) = 1$.

Homotopy invariance. In this section we will show that the Brouwer degree does not depend on the choice of regular values and is invariant under smooth homotopy.

Lemma 3.7.2. Let $M$ be the boundary of a compact oriented manifold $X$, oriented as the boundary of $X$. If $f : M \to N$ extends to a smooth map $F : X \to N$ then $\deg(f, y) = 0$ for every regular value $y$.

Proof. (a) Assume that $y$ is a regular value of $F$. Then $F^{-1}(y)$ is a 1-dimensional manifold of dimension 1 whose boundary is $F^{-1}(y) \cap M = f^{-1}(y)$, by Theorem 3.4.8.

By the Classification of one-dimensional manifolds, $F^{-1}(y)$ is the disjoint union of arcs and circles. Let $A$ be a component that intersects $M$. Then $A$ is an arc with boundary $\{a, b\} \subset M$.

We will show that $\text{sign}(\det(df_a)) = -\text{sign}(\det(df_b))$. Taking sum over all arc components of $F^{-1}(y)$ would give us $\deg(f, y) = 0$.

\cite{Brouwer} L. E. J. Brouwer (1881–1966) is a Dutch mathematician. He had many important contributions in the early development of topology, and founded Intuitionism.
An orientation on $A$. Let $x \in A$. Recall that $TA_x$ is the kernel of $dF_x : TX_x \to TN_y$. We will choose the orientation on $TA_x$ such that this orientation together with the pull-back of the orientation of $TN_y$ via $dF_x$ is the orientation of $X$. Let $(v_2, v_3, \ldots, v_{n+1})$ be a positive basis for $TN_y$. Let $v_1 \in TA_x$ such that $\{v_1, dF_x^{-1}(v_2), \ldots, dF_x^{-1}(v_{n+1})\}$ is a positive basis for $TX_x$. Then $v_1$ determine the positive orientation on $TA_x$.

At $x = a$ or at $x = b$ we have $df_x = df_x|TM_a$. Therefore $df_x$ is orientation-preserving on $TM_x$ oriented by the basis $\{df_x^{-1}(v_2), \ldots, df_x^{-1}(v_{n+1})\}$.

We claim that exactly one of the two above orientations of $TM_x$ at $x = a$ or $x = b$ is opposite to the orientation of $TM_x$ as the boundary of $X$. This would show that $\text{sign}(\text{det}(d_f)) = - \text{sign}(\text{det}(d_f))$.

Observe that if at $a$ the orientation of $TA_a$ is pointing outward with respect to $X$ then $b$ the orientation of $TA_b$ is pointing inward, and vice versa. Indeed, since $A$ is a smooth arc it is parametrized by a smooth map $\gamma(t)$ such that $\gamma(0) = a$ and $\gamma(1) = b$. If we assume that the orientation of $TA_{\gamma(t)}$ is given by $\gamma'(t)$ then it is clear that at $a$ the orientation is inward and at $b$ it is outward.

(b) Suppose now that $y$ is not a regular value of $F$. There is a neighborhood of $y$ in the set of all regular values of $f$ such that $\text{deg}(f, z)$ does not change in this neighborhood. Let $z$ be a regular value of $F$ in this neighborhood, then $\text{deg}(f, z) = \text{deg}(F, z) = 0$ by (a), and $\text{deg}(f, z) = \text{deg}(f, y)$. Thus $\text{deg}(f, y) = 0$. 

**Lemma 3.7.3.** If $f$ is smoothly homotopic to $g$ then $\text{deg}(f, y) = \text{deg}(g, y)$ for any common regular value $y$.

**Proof.** Let $I = [0, 1]$ and $X = M \times I$. Since $f$ be homotopic to $g$ there is a smooth map $F : X \to N$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

The boundary of $X$ is $(M \times \{0\}) \cup (M \times \{1\})$. Then $F$ is an extension of the pair $f, g$ from $\partial X$ to $X$, thus $\text{deg}(F|_{\partial X}, y) = 0$ by the above lemma.

Note that one of the two orientations of $M \times \{0\}$ or $M \times \{1\}$ as the boundary of $X$ is opposite to the orientation of $M$ (this is essentially for the same reason as in the proof of the above lemma). Therefore $\text{deg}(F|_{\partial X}, y) = \pm(\text{deg}(f, y) - \text{deg}(g, y)) = 0$, so $\text{deg}(f, y) = \text{deg}(g, y)$. 

**Lemma 3.7.4 (Homogeneity of manifold).** Let $N$ be a connected boundaryless manifold and let $y$ and $z$ be points of $N$. Then there is a self diffeomorphism $h : N \to N$ that is smoothly isotopic to the identity and carries $y$ to $z$.

We do not present a proof for this lemma. The reader can find a proof in [Mil97] p. 22.

**Theorem 3.7.5.** Let $M$ and $N$ be boundaryless, oriented manifolds of the same dimensions. Further suppose that $M$ is compact and $N$ is connected. The Brouwer degree of a map from $M$ to $N$ does not depend on the choice of regular values and is invariant under smooth homotopy.

Therefore from now on we will write $\text{deg}(f)$ instead of $\text{deg}(f, y)$. 
3.7. BROUWER DEGREE

**Proof.** We have already shown that degree is invariant under homotopy.

Let \( y \) and \( z \) be two regular values for \( f : M \to N \). Choose a diffeomorphism \( h \) from \( N \) to \( N \) that is isotopic to the identity and carries \( y \) to \( z \).

Note that \( h \) preserves orientation. Indeed, there is a smooth isotopy \( F : N \times [0, 1] \to N \) such that \( F_0 = h \) and \( F_1 = \text{id} \). Let \( x \in M \), and let \( \varphi : \mathbb{R}^n \to M \) be an orientation-preserving parametrization of a neighborhood of \( x \) with \( \varphi(0) = x \). Since \( dF_t(x) \circ d\varphi_0 : \mathbb{R}^n \times \mathbb{R} \) is smooth with respect to \( t \), the sign of \( dF_t(x) \) does not change with \( t \).

As a consequence, \( \deg(f, y) = \deg(h \circ f, h(y)) \).

Finally since \( h \circ f \) is homotopic to \( \text{id} \circ f \), we have \( \deg(h \circ f, h(y)) = \deg(\text{id} \circ f, h(y)) = \deg(f, h(y)) = \deg(f, y) \). \( \square \)

**Example 3.7.6.** Let \( M \) be a compact, oriented and boundaryless manifold. Then the degree of the identity map on \( M \) is 1. On the other hand the degree of a constant map on \( M \) is 0. Therefore the identity map is not homotopic to a constant map.

**Example 3.7.7 (Proof of the Brouwer fixed point theorem via the Brouwer degree).** We can prove that \( D^{n+1} \) cannot retract to its boundary (this is [3.5.3 for the case of \( D^{n+1} \)) as follows. Suppose that there is such a retraction, a smooth map \( f : D^{n+1} \to S^n \) that is the identity on \( S^n \). Define \( F : [0, 1] \times S^n \) by \( F(t, x) = f(tx) \). Then \( F \) is a smooth homotopy from a constant map to the identity map on the sphere. But these two maps have different degrees.

**Theorem 3.7.8 (The fundamental theorem of Algebra).** Any non-constant polynomial with real coefficients has at least one complex root.

**Proof.** Let \( p(z) = z^n + a_1z^{n-1} + a_2z^{n-2} + \cdots + a_{n-1}z + a_n \) with \( a_i \in \mathbb{R} \), \( 1 \leq i \leq n \). Suppose that \( p \) has no root, that is, \( p(z) \neq 0 \) for all \( z \in \mathbb{C} \). As a consequence, \( a_n \neq 0 \).

For \( t \in [0, 1] \), let

\[
q_t(z) = (1 - t)^nz^n + a_1(1 - t)^{n-1}tz^{n-1} + \cdots + a_{n-1}(1 - t)t^{n-1}z + a_n t^n.
\]

Then \( q_t(z) \) is continuous with respect to the pair \((t, z)\). Notice that if \( t \neq 0 \) then \( q_t(z) = t^n p((1 - t)t^{-1}z) \), and \( q_0(z) = z^n \) while \( q_1(z) = a_n \).

If we restrict \( z \) to the set \( \{ z \in \mathbb{C} \mid |z| = 1 \} = S^1 \) then \( q_t(z) \) has no roots, so \( \frac{q_t(z)}{|q_t(z)|} \) is a continuous homotopy of maps from \( S^1 \) to itself, starting with the polynomial \( z^n \) and ending with the constant polynomial \( \frac{a_n}{|a_n|} \). But these two polynomials have different degrees, a contradiction. \( \square \)

**Vector field.**

**Definition 3.7.9.** A (smooth) (tangent) vector field on a boundaryless manifold \( M \subset \mathbb{R}^k \) is a smooth map \( v : M \to \mathbb{R}^k \) such that \( v(x) \in TM_x \) for each \( x \in M \).
Example 3.7.10. Let \( v : S^1 \to \mathbb{R}^2, v((x, y)) = (-y, x) \), then it is a nonzero (not zero anywhere) tangent vector field on \( S^1 \).

Similarly we can find a nonzero tangent vector field on \( S^n \) with odd \( n \).

Theorem 3.7.11 (The Hairy Ball Theorem). There is no smooth nonzero tangent vector field on \( S^n \) if \( n \) is even. \(^2\)

**Proof.** Suppose that \( v \) is a nonzero tangent smooth vector field on \( S^n \). Let \( w(x) = \frac{v(x)}{|v(x)|} \), then \( w \) is a unit smooth tangent vector field on \( S^n \).

Notice that \( w(x) \) is perpendicular to \( x \). On the plane spanned by \( x \) and \( w(x) \) we can easily rotate vector \( x \) to vector \(-x\). Precisely, let \( F_t(x) = \cos(t) \cdot x + \sin(t) \cdot w(x) \) with \( 0 \leq t \leq \pi \), then \( F \) is a homotopy from \( x \) to \(-x\). But the degrees of these two maps are different, see 3.7.20. \(\square\)

**Problems.**

3.7.12. Find the topological degree of a polynomial on \( \mathbb{R} \). Notice that although the domain \( \mathbb{R} \) is not compact, the topological degree is well-defined for polynomial.

3.7.13. Let \( f : S^1 \to S^1, f(z) = z^n \), where \( n \in \mathbb{Z} \). We can also consider \( f \) as a vector-valued function \( f : \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (f_1(x, y), f_2(x, y)) \). Then \( f = f_1 + if_2 \).

1. Recalling the notion of complex derivative and the Cauchy-Riemann condition, check that \( \det(f_z) = |f'(z)|^2 \).
2. Check that all values of \( f \) are regular.
3. Check that \( \deg(f, y) = n \) for all \( y \in S^1 \).

3.7.14. Show that \( \deg(f, y) \) is locally constant on the subspace of all regular values of \( f \).

3.7.15. What happens if we drop the condition that \( N \) is connected in Theorem 3.7.5? Where do we use this condition?

3.7.16. Let \( M \) and \( N \) be oriented boundaryless manifolds, \( M \) is compact and \( N \) is connected. Let \( f : M \to N \). Show that if \( \deg(f) \neq 0 \) then \( f \) is onto.

3.7.17. * Let \( r_1 : S^n \to S^n \) be the reflection map

\[
r_1((x_1, x_2, \ldots, x_i, \ldots, x_{n+1})) = (x_1, x_2, \ldots, -x_i, \ldots, x_{n+1}).
\]

Compute \( \deg(r_1) \).

3.7.18. * Let \( f : S^n \to S^n \) be the map that interchanges two coordinates:

\[
f((x_1, x_2, \ldots, x_i, \ldots, x_{n+1})) = (x_1, x_2, \ldots, x_i, \ldots, x_{n+1}).
\]

Compute \( \deg(f) \).

3.7.19. Suppose that \( M, N, P \) are compact, oriented, connected, boundaryless \( m \)-manifolds. Let \( M \xrightarrow{f} N \xrightarrow{g} P \). Then \( \deg(g \circ f) = \deg(f) \deg(g) \).

3.7.20. * Let \( r : S^n \to S^n \) be the antipodal map

\[
r((x_1, x_2, \ldots, x_{n+1})) = (-x_1, -x_2, \ldots, -x_{n+1}).
\]

Compute \( \deg(r) \).

\(^2\)This result can be interpreted cheerfully as that on our head there must be a spot with no hair!
3.7.21. Let \( f : S^4 \to S^4 \), \( f((x_1, x_2, x_3, x_4, x_5)) = (x_2, x_4, -x_1, x_5, -x_3) \). Find \( \deg(f) \).

3.7.22. Show that any polynomial of degree \( n \) gives rise to a map from \( S^2 \) to itself of degree \( n \).

3.7.23. Find a map from \( S^2 \) to itself of any given degree.

3.7.24. If \( f, g : S^n \to S^n \) be smooth such that \( ||f(x) - g(x)|| < 2 \) for all \( x \in S^n \) then \( f \) is smoothly homotopic \( g \).

   \[ \text{Hint:} \quad \text{Note that } f(x) \text{ and } g(x) \text{ will not be antipodal points. Use the homotopy} \]
   \[ f_t(x) = \frac{(1 - t)f(x) + tg(x)}{||(1 - t)f(x) + tg(x)||} \]

3.7.25. If \( f, g : S^n \to S^n \) be smooth such that \( f(x) \neq g(x) \) for all \( x \in S^n \) then \( f \) is smoothly homotopic to \(-g\).

3.7.26. Let \( f : M \to S^n \) be smooth. Show that if \( \dim(M) < n \) then \( f \) is homotopic to a constant map.

   \[ \text{Hint:} \quad \text{Using Sard Theorem show that } f \text{ cannot be onto.} \]

3.7.27 (Brouwer fixed point theorem for the sphere). Let \( f : S^n \to S^n \) be smooth. If \( \deg(f) \neq (-1)^{n+1} \) then \( f \) has a fixed point.

   \[ \text{Hint:} \quad \text{If } f \text{ does not have a fixed point then } f \text{ will be homotopic to the reflection map.} \]

3.7.28. Show that any map of from \( S^n \) to \( S^n \) of odd degree carries some pair of antipodal points to a pair of antipodal point.
Guide for Further Reading in Differential Topology

We have closely followed John Milnor’s masterpiece [Mil97]. Another excellent text is [GP74]. There are not many textbooks such as these two books, presenting differential topology to undergraduate students.

The book [Hir76] is a technical reference for some advanced topics. The book [DFN85] is a masterful presentation of modern topology and geometry, with some enlightening explanations, but it sometimes requires knowledge of many topics. The book [Bre93] is rather similar in aim, but is more like a traditional textbook.

An excellent textbook for differential geometry of surfaces is [dC76].
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vector field

Zorn lemma
You may say I’m a dreamer
But I’m not the only one
I hope someday you’ll join us ...

John Lennon, Imagine.